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I-B-VP: Closing the spatial-product extension ledger

Vacuum polarization, stress tensor, and induced current on a prescribed conical spacetime with Aharonov–Bohm holonomy

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Abstract

We give a unified construction and extension ledger for a massless scalar field on the regular conical product prescribed in I-B. The spatial operator, its self-adjoint extension, the state prescription, the curvature coupling, and the flat-bundle holonomy are explicit inputs. The extension class consists of transverse self-adjoint extensions tensored with the ordinary longitudinal Laplacian; it includes rotation-breaking finite-dimensional boundary conditions but excludes general non-product extensions.

For the Friedrichs extension, an explicit heat-kernel estimate proves infrared finiteness of the massless spectral covariance on compact tests. Direct Euclidean subtraction leaves a jointly real-analytic bisolution at every regular diagonal point, including image-boundary parameters. Analytic continuation gives the local Hadamard form used for point splitting.

For one complex field, with $\Phi = (\varphi_1 + \mathrm{i}\varphi_2)/\sqrt{2}$, $T = T[\varphi_1] + T[\varphi_2]$, signature $(-,+,+,+)$, and mixed components ordered as (t, r, φ, z), define

$$P_\beta = \alpha^{-2}(1 - 6\beta(1-\beta)) - 1, \qquad C_\beta = \alpha^{-4}(1 - 30\beta^2(1-\beta)^2) - 1.$$

The Friedrichs results are

$$\langle \Phi^\dagger \Phi \rangle_{\mathrm{ren}} = \frac{P_\beta}{48\pi^2 r^2}, \qquad \langle j_\varphi \rangle_{\mathrm{ren}} = -\frac{e\,\beta(1-\beta)(1-2\beta)}{12\pi^2\alpha^2 r^2},$$

$$\langle T^{\mu}{}_{\nu} \rangle_{\mathrm{ren}} = \frac{C_\beta}{720\pi^2 r^4}\mathrm{diag}\,(1,1,-3,1) + 2\left(\xi - \frac{1}{6}\right)\frac{P_\beta}{48\pi^2 r^4}\mathrm{diag}\,(4,-2,6,4).$$

The neutral real tensor is half the untwisted complex tensor. The current uses the complex first jet of the regular two-point function; replacing that function by its real part would erase it.

For every nonnegative extension in the declared product class, a finite-rank Krein formula gives an infrared-finite spectral state and an analytic regular-part correction. We supply the formula also on boundary charts of the extension space. Positivity is equivalent to positivity of the finite boundary relation. Negative transverse eigenvalues produce unstable longitudinal bands; a stationary finite-second-moment state is then excluded by a commutator-growth argument. Two projected alternatives are recorded, together with their nonlocality and Hadamard defects.

Scale-invariant irregular modes admit closed forms expressed through analytic branches of the Friedrichs polynomials at shifted arguments. At half holonomy an additional $\mathbb{CP}^1$ family has explicit angular-dependent observables. Scale-dependent nonnegative extensions have convergent integral representations and proved asymptotics. Generic rotation-breaking extensions with unequal deficient orders can carry a radial current.

The familiar Friedrichs coefficients are not claimed as new. Historical formula comparisons and all inherited numerical residuals are retained with their evidence grades. No numerical execution, apex renormalization, backreaction, interacting construction, or downstream programme closure is asserted.

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Book expansion: umbrella version normalization

This book edition normalizes the consolidation to **version 4**. Do not cite it as ``version 1" in one place and ``version 3" elsewhere. Historical audit grades are retained as documentary records and are *not* independently re-executed. AB/SAE companion records are kept as separate versioned entries in the bibliography.

Purpose, provenance, and the consolidation

Vacuum polarization on an idealized cosmic-string background is established subject matter [1, 2, 3, 4, 6]. The separate question of a boundary condition at the removed apex is essential: the regular metric alone does not select a self-adjoint extension [11, 12].

This manuscript consolidates the supplied records I-B-VP-v2.tex, I-B-VP-AB.tex, and I-B-VP-SAE.tex [36, 37, 38]. Those files are retained records; this consolidation neither deletes them nor assumes that their version names form an unambiguous numerical sequence. The first supplied file already identifies itself as version 2, whereas the AB and SAE records refer to an earlier I-B-VP.tex as version 1. We therefore identify predecessor records by

filename.

How to read this version. Sections – contain the Friedrichs construction and all three observables. Sections – treat the product extension ledger. Section is the unified status table. The appendices preserve the reported computations and corrections without using them as proof.

Preserved additions: the AB change record

The AB source's items C1–C7 are retained in substance:

1. The induced current is added to the Wick square and stress tensor.

2. Explicit comparisons with the flux literature are included.

3. Reports A1–A5 include character-weighted quotient images and high-precision contour comparisons.

4. The Friedrichs formulas hold for every $0 < \alpha < 1$, not only quotient angles. Their analytical proof does not use finite sampling.

5. The AB source revises its explanation of the earlier 10^{-17}-scale residuals: it reports a cast of image terms to IEEE double complex and an omitted diffraction term at a rounded quotient parameter. These remain reported diagnostics because the scripts and raw logs were not supplied.

6. The bibliography retains the source's access grades, with their meaning stated in Section . The Smith chapter's 1989/1990 citation variation is retained as a bibliographic note.

7. The deficient twisted modes and their extension families are made explicit.

Preserved additions: the SAE change record

The SAE source's items D1–D6 are likewise retained:

1. Boundary data, a Krein correction, positivity, unstable extensions, projected constructions, scale-invariant closed forms, and scale-dependent integral representations are included.

2. The AKO and Filgueiras–Moraes comparisons and the SAE verification reports are retained.

3. Nonnegative non-Friedrichs product extensions move from ``Open'' in the predecessor ledger to constructed spectral states with evaluated observables or convergent evaluation formulas.

4. The established Friedrichs formulas are unchanged. Shifted arguments refer to an entire *geometric branch*, not to the globally periodic physical absolute-value mode sum.

5. Rotation-breaking $U(2)$ boundary conditions are admitted only after dropping rotation invariance, while retaining the spatial-product assumption.

6. Bibliographic access marks remain provenance statements; they are not a replacement for a formula-level comparison or an execution certificate.

Analytical corrections required for a coherent merge

A submission cannot preserve a false implication merely because it appeared in a predecessor. The following corrections are incorporated and their affected reports remain visible:

1. Infrared finiteness is proved. Nonnegativity alone is not used to infer a massless covariance. The Friedrichs heat estimate and the non-Friedrichs Krein bound supply the necessary estimates.

2. Image boundaries are included. The total Euclidean remainder is treated by joint ellipticity. No diagonal parameter is discarded as a measure-zero exception.

3. Current subtraction is stated correctly. The derivative difference of the direct parametrix does not vanish along arbitrary approaches to coincidence. The current is defined using the subtracted complex kernel. For the time splitting used to compute its azimuthal component, the direct contribution is identically zero.

4. Boundary charts are genuine boundary relations. An ``infinite eigenvalue'' is replaced by an explicit subspace and a compressed Hermitian relation. The finite-matrix small-radius limit is not asserted for every boundary chart.

5. Small-radius remainder orders are corrected. The first correction for $0 < \nu < 1/2$ is retained, but its remainder has additional regimes at $\nu = 1/3$. The logarithmic case $\nu = 1/2$ is supplied.

6. Asymptotic zeros are respected. A relative error is not asserted when a displayed leading stress coefficient vanishes.

7. Transverse and full spectra are separated. A negative transverse eigenvalue generates a full longitudinal continuous band, not a full-space L^2 eigenfunction.

8. State selection is not universal state uniqueness. The distinguished zero-mean spectral Fock state is specified. No classification of all algebraic ground states is claimed.

9. Projection changes the field algebra. The two projected constructions are not states with the original local canonical commutator. Their defects are stated on their actual domains.

10. Computation is not silently promoted to proof. All inherited execution claims are and as rigorous total-error bounds.

These corrections do not alter the Friedrichs polynomials, its current, the scale-invariant correction formulas, or the finite-chart Krein coefficient.

Background, non-extraction, and mathematical dependence

The Enter companion host is prescribed, not derived in this paper. I-B distinguishes a finite residual density matrix from a covariant tensor-valued distribution: the field, observable, state, renormalization prescription, and distributional interpretation must be supplied separately [41]. We supply those ingredients for a free test field. None is extracted from an Enter readout.

The destructive-readout interpretation of Enter [39, 40, 49] and the reporting discipline of XVI-O1-ST [47] constrain interpretation but do not construct the continuum covariance. The finite instrument and its limitation are reproduced in Section .

Use of I-B-G. The supplied Friedrichs foundation identifies I-B-G-v2.tex, with source labels ass:host, ass:product, thm:positivity, thm:green, thm:speed, thm:closed, cor:structure, lem:window, thm:transfer, and ass:transfer-contract [42]. The AB and SAE records also refer to an earlier title and theorem numbering. This

paper uses source labels when discussing the companion and reproduces the operator, kernel, boundary, and positivity calculations needed for its principal results. Finite propagation and supported transfer are used only in their separately stated scope.

Companion comparisons are not imported constructions. X-A-ST, X-B-ST, and X-C-ST concern supported scalar completion operations [43, 44, 45]. XI-E separates a named Hadamard product domain from selection of a positive state [46]. Oai282-A separates finite tests from continuum coverage [48]. These comparisons do not import a gauge domain, interacting product, contact classification, or dynamics law.

Meaning of ``Open''. An item marked **Open** is not resolved by this manuscript. This is not a claim that it is unknown in the literature.

Host, bundle, field normalization, and signature

Assumption (Prescribed background). Fix

$$0 < \mu < 2\pi, \qquad \alpha = 1 - \frac{\mu}{2\pi} \in (0, 1).$$

The regular spacetime is

$$M = \mathbb{R}_t \times (0, \infty)_r \times (\mathbb{R}/2\pi\mathbb{Z})_\varphi \times \mathbb{R}_z,$$

with

$$g = -\mathrm{d}t^2 + \mathrm{d}r^2 + \alpha^2 r^2 \mathrm{d}\varphi^2 + \mathrm{d}z^2.$$

The metric completion adds $\mathscr{W} = \{r = 0\}$, but $\mathscr{W} \subset M$. We use $\hbar = c = 1$, and t is length-valued time.

The massless equation is

$$P\Phi = (-\mathrm{Box}_g + \xi R)\Phi = (\partial_t^2 + B)\Phi = 0.$$

On M, $R = 0$. No distributional apex curvature coupling is included.

Let

$$N = (0,\infty) \times S^1 \times \mathbb{R}, \qquad q = \mathrm{d}r^2 + \alpha^2 r^2 \mathrm{d}\varphi^2 + \mathrm{d}z^2, \qquad \mathrm{d}V = \alpha r \, \mathrm{d}r \, \mathrm{d}\varphi \, \mathrm{d}z.$$

For $\beta \in [0,1)$, the flat Hermitian line bundle L_β is represented on the angular cover by

$$u(r, \varphi + 2\pi, z) = e^{2\pi i \beta} u(r, \varphi, z).$$

In a lifted parallel frame its connection is ordinary differentiation. Put

$$q_n = n + \beta, \qquad \nu_n = \frac{|q_n|}{\alpha}, \qquad \psi_n(\varphi) = \frac{e^{\mathrm{i} q_n \varphi}}{\sqrt{2\pi\alpha}}.$$

The spatial Hilbert space is $\mathscr{H}_\beta = L^2_\beta(N, \mathrm{d}V)$.

Assumption (Extension class). The main ledger concerns

$$B = B_\perp \otimes \mathrm{id} + \mathrm{id} \otimes (-\partial_z^2),$$

where $B_\perp$ is a self-adjoint extension of the transverse cone Laplacian in $L^2_\beta(C^\circ_\alpha, \alpha r \, \mathrm{d}r \, \mathrm{d}\varphi)$. It is complex linear.

Rotation invariance is imposed for the diagonal subledger but not for the full finite-dimensional boundary relation.

General z-dependent, longitudinal-momentum-dependent, and non-product extensions are outside this class.

The Friedrichs extension is denoted B_F. It is the closure of the nonnegative energy form initially defined on compactly supported smooth sections.

Real and complex normalization

A neutral real field uses $\beta = 0$. For the charged field,

$$\Phi = \frac{\varphi_1 + \mathrm{i}\varphi_2}{\sqrt{2}}, \qquad T^{\mathrm{complex}} = T[\varphi_1] + T[\varphi_2].$$

Thus $\langle \Phi^\dagger \Phi \rangle$ has the same scalar covariance normalization as one real Wick square, whereas the complex stress tensor has twice the corresponding real differential expression. The real components need not be separate globally trivial bundles.

The antiparticle sector uses the conjugate bundle and conjugate extension. A real sign bundle at $\beta = 1/2$ is not what

is called one charged complex field here.

For

$$D_{\mu} = \nabla_{\mu} + \mathrm{i}eA_{\mu},$$

a periodic gauge may be chosen locally with $eA = \beta\,\mathrm{d}\varphi$, up to an integer gauge shift. The twisted parallel representation satisfies

$$\Phi_{\mathrm{tw}} = e^{\mathrm{i}\beta\varphi}\Phi_{\mathrm{per}}.$$

For a flux $\mathscr{F}$, our convention is $\beta = \mathrm{frac}\,(e\mathscr{F}/2\pi)$.

Signature ledger

On a developing chart,

$$X = r\cos(\alpha\varphi), \qquad Y = r\sin(\alpha\varphi).$$

If a companion completion uses

$$\eta = \mathrm{d}t^2 - \mathrm{d}X^2 - \mathrm{d}Y^2 - \mathrm{d}z^2,$$

then

$$\mathrm{dev}^{*}\eta = -g, \qquad -\mathrm{Box}_g = \mathrm{dev}^{*}\mathrm{Box}_{\eta}.$$

The equation operator and action Hessian have opposite signs. This distinction matters when comparing inverse conventions.

Euclidean continuation uses

$$g_E = \mathrm{d}\tau^2 + \mathrm{d}r^2 + \alpha^2 r^2 \mathrm{d}\varphi^2 + \mathrm{d}z^2, \qquad L_E = -\partial_{\tau}^2 + B.$$

With $t = -\mathrm{i}\tau$, mixed stress components obey

$$T^{t}{}_{t} = T^{\tau}{}_{\tau}\big|_E, \qquad T^{i}{}_{j} = T^{i}{}_{j}\big|_E.$$

The static energy density is

$$\varepsilon = T_{tt} = -T^{t}{}_{t}.$$

Angular lifts and distances

For chosen lifts define

$$\theta = \alpha(\varphi - \varphi'), \qquad \mathcal{K}(\theta) = \{k \in \mathbb{Z} : |\theta - 2\pi\alpha k| < \pi\}.$$

Set

$$R_k^2 = r^2 + r'^2 - 2rr'\cos(\theta - 2\pi\alpha k) + \Delta z^2,$$

$$\rho(u)^2 = r^2 + r'^2 + 2rr'\cosh u + \Delta z^2,$$

$$\rho_0^2 = (r + r')^2 + \Delta z^2.$$

An image boundary is a locus

$$|\theta - 2\pi\alpha k| = \pi.$$

If $\theta = \psi + 2\pi\alpha\ell$, with $\psi \in (-\pi\alpha, \pi\alpha]$, a lifted kernel satisfies

$$K(\theta) = e^{2\pi i\beta\ell}K(\psi).$$

A reduced-angle expression must retain this bundle factor. All diagonal calculations use one local parallel frame.

Friedrichs heat kernel and the massless state

Proposition (Heat kernel and large-time estimate). *Relative to* dV*, the Friedrichs heat kernel is*

$$K_\beta(s; x, x') = \frac{e^{-\Delta z^2/(4s)}}{\sqrt{4\pi s}} \frac{e^{-(r^2 + r'^2)/(4s)}}{4\pi\alpha s} \sum_{n \in \mathbb{Z}} e^{iq_n(\varphi - \varphi')} I_{\nu_n}\left(\frac{rr'}{2s}\right).$$

For each compact regular spatial set K*,*

$$\sup_{x, x' \in K} |K_\beta(s; x, x')| \le C_K s^{-3/2 - \nu_*}, \qquad s \ge 1, \qquad \nu_* = \frac{\min(\beta, 1 - \beta)}{\alpha}.$$

At $\beta = 0$, $v_* = 0$.

Proof. The Friedrichs radial transform uses $J_\nu(kr)$ with measure $k\,\mathrm{d}k$. Weber's integral is

$$\int_0^\infty e^{-sk^2} J_\nu(kr) J_\nu(kr')k\,\mathrm{d}k = \frac{e^{-(r^2+r'^2)/(4s)}}{2s} I_\nu\left(\frac{rr'}{2s}\right).$$

Combining the normalized angular modes with the longitudinal heat kernel gives [22, 24].

On K, the power series gives, uniformly for $s \geq 1$,

$$I_\nu\left(\frac{rr'}{2s}\right) \leq C_K \frac{(C_K/s)^\nu}{\Gamma(1+\nu)}.$$

The orders grow linearly in $|n|$, so their sum is bounded by $C_K s^{-\nu_*}$. The remaining prefactor is $O(s^{-3/2})$.

Lemma (Infrared finiteness). *For every* $v \in C_c^\infty(N; L_\beta)$, $B_F^{-1/4} v \in \mathscr{H}_\beta$. *On a fixed compact support this map is continuous in the compact smooth test topology. There is no* L^2 *zero eigenfunction.*

Proof. The spectral theorem gives

$$\lVert B_F^{-1/4} v \rVert^2 = \frac{1}{\sqrt{\pi}} \int_0^\infty s^{-1/2} \langle v, e^{-sB_F} v \rangle\,\mathrm{d}s.$$

For $s \leq 1$, contraction bounds the integrand by $s^{-1/2}\|v\|^2$. For $s \geq 1$, Proposition bounds it by

$$C_K s^{-2-\nu_*} \|v\|_{L^1}^2.$$

Both integrals converge and give continuity. Zero form energy would imply a locally parallel section; no nonzero parallel section is square integrable on N.

Definition (Distinguished spectral state). For a nonnegative spatial extension whose compact tests belong to $\mathrm{dom}(B^{-1/4})$, define

$$K_B f = \int_{\mathbb{R}} e^{\mathrm{i}t\sqrt{B}} (2\sqrt{B})^{-1/2} f(t)\,\mathrm{d}t,$$

and the positive-frequency kernel

$$W_B(t-t';x,x') = \left[\frac{e^{-\mathrm{i}(t-t')\sqrt{B}}}{2\sqrt{B}}\right](x,x').$$

The zero-mean quasifree spectral Fock state has smeared covariance $\langle K_B f, K_B g\rangle$.

For a complex field the particle and antiparticle constructions use B and its conjugate extension. Charge-changing two-point functions vanish.

The algebra is the extension-selected test-field CCR algebra. Its commutator is the causal kernel of that extension. No compact-test exactness assertion $\ker E = PC_c^\infty(M)$ is needed. That separate question is recorded as open in the supplied I-B-G rem:compact-exact.

Proposition (Friedrichs state and symmetries). *The Friedrichs prescription is a continuous positive covariance on compact smooth spacetime tests and defines a spectral ground state. With*

$$E_\beta(t) = \frac{\sin(t\sqrt{B_F})}{\sqrt{B_F}},$$

the real and charged commutators are respectively

$$[\varphi(X),\varphi(X')] = -\mathrm{i}E_0(X,X'), \qquad [\Phi(X),\Phi^\dagger(X')] = -\mathrm{i}E_\beta(X,X').$$

The state is invariant under t,z translations, rotations, and proper time-oriented (t,z) boosts. Its covariance has scaling degree two:

$$W_F(\lambda t,\lambda r,\varphi,\lambda z;\lambda t',\lambda r',\varphi',\lambda z') = \lambda^{-2}W_F(X,X').$$

Proof. Lemma bounds the one-particle vector on fixed compact supports. The Gram form proves positivity and continuity; the spectral Fock construction gives a nonnegative Hamiltonian [30]. The difference of time orientations is

$$\frac{e^{-\mathrm{i}t\sqrt{B}} - e^{\mathrm{i}t\sqrt{B}}}{2\sqrt{B}} = -\mathrm{i}\frac{\sin(t\sqrt{B})}{\sqrt{B}}.$$

Differentiating in the second time variable at equal times gives the canonical +iid sign.

Translations and rotations commute with the operator. For boosts use

$$W_F = \frac{1}{2\pi\alpha}\sum_n e^{\mathrm{i}q_n(\varphi-\varphi')}\int_0^\infty k\,\mathrm{d}k\,J_{\nu_n}(kr)J_{\nu_n}(kr')$$

$$\times\int_{\mathbb{R}}\frac{\mathrm{d}p}{2\pi}\frac{e^{\mathrm{i}p\Delta z-\mathrm{i}\sqrt{k^2+p^2}\Delta t}}{2\sqrt{k^2+p^2}}.$$

The inner mass-shell measure is the boost-invariant 1 + 1-dimensional positive-frequency measure of mass k. The infrared estimate justifies compact testing. Rescaling k, p proves the dilation identity.

For Friedrichs, intrinsic finite propagation gives local commutativity for intrinsically spacelike-separated supports [42, thm:speed]. This is not a statement about an arbitrary completion.

Remark (Extension selection is not state uniqueness). The zero-mean spectral state is distinguished by the prescription, not by a theorem classifying every algebraic ground state. Additional zero-frequency covariance terms or appropriate coherent shifts are not ruled out merely by positivity of the extension.

Euclidean kernels, diffraction, and local Hadamard form

For Euclidean points set

$$\cosh\eta=\frac{r^2+r'^2+\Delta\tau^2+\Delta z^2}{2rr'},\qquad \sigma_E^2=r^2+r'^2-2rr'\cos\theta+\Delta\tau^2+\Delta z^2.$$

Proposition (Euclidean mode sum). *The heat-integrated kernel*

$$G_{F,E}(X,X')=\int_0^\infty\frac{e^{-\Delta\tau^2/(4s)}}{\sqrt{4\pi s}}K_\beta(s;x,x')\,\mathrm{d}s$$

is a fundamental distribution for L_E in each variable. For $\eta>0$,

$$G_{F,E}=\frac{1}{8\pi^2\alpha rr'\sinh\eta}\sum_{n\in\mathbb{Z}}e^{-|q_n|\eta/\alpha}e^{\mathrm{i}q_n(\varphi-\varphi')}$$

$$=\frac{1}{8\pi^2\alpha rr'\sinh\eta}\left[\frac{e^{-\beta a+\mathrm{i}\beta b}}{1-e^{-a+\mathrm{i}b}}+\frac{e^{-(1-\beta)a-\mathrm{i}(1-\beta)b}}{1-e^{-a-\mathrm{i}b}}\right],$$

where $a=\eta/\alpha$, $b=\varphi-\varphi'$. For $\beta=0$,

$$G_{F,E} = \frac{\sinh(\eta/\alpha)}{8\pi^2 \alpha r r' \sinh\eta[\cosh(\eta/\alpha) - \cos(\varphi - \varphi')]}.$$

At $\alpha = 1, \beta = 0$, *this is* $(4\pi^2\sigma_E^2)^{-1}$.

Proof. Large-time convergence follows from Proposition ; the additional Euclidean factor gives $s^{-2-\nu_*}$. Integrating the heat equation proves the fundamental-distribution identities. Substitute $\zeta = rr'/(2s)$ and use

$$\int_0^\infty e^{-\zeta\cosh\eta} I_\nu(\zeta)\,\mathrm{d}\zeta = \frac{e^{-\nu\eta}}{\sinh\eta}, \qquad \eta > 0, \quad \nu > -1.$$

The constants become $(8\pi^2\alpha r r')^{-1}$. Absolute convergence justifies the Friedrichs sum for $\eta > 0$. Splitting $n \geq 0$ and $n \leq -1$ gives the two geometric series [22, 24].

Angular summation

Define

$$\Phi_\beta(u,\theta) = \sum_{n\in\mathbb{Z}} \sin(\pi\nu_n) e^{-\nu_n u} e^{\mathrm{i}q_n\theta/\alpha}, \qquad u > 0.$$

Lemma (Sommerfeld summation and boundary control). *For* $\zeta > 0$, *at generic angles,*

$$\sum_n I_{\nu_n}(\zeta) e^{\mathrm{i}q_n\theta/\alpha} = \alpha \sum_{k\in\mathcal{K}(\theta)} e^{2\pi\mathrm{i}\beta k} e^{\zeta\cos(\theta - 2\pi\alpha k)} - \frac{1}{\pi}\int_0^\infty e^{-\zeta\cosh u}\Phi_\beta(u,\theta)\,\mathrm{d}u.$$

The identity extends after joint distributional testing through image boundaries. Near such a boundary θ_j,

$$|\Phi_\beta(u,\theta)| \leq C\Big(1 + [u^2 + (\theta - \theta_j)^2]^{-1/2}\Big).$$

The weight decays exponentially at large u. For $\beta = 0$,

$$\Phi_0(u,\theta) = \frac{1}{2}\sum_{\pm} \frac{\sin((\pi \pm \theta)/\alpha)}{\cosh(u/\alpha) - \cos((\pi \pm \theta)/\alpha)}.$$

Proof. Set $A = \pi/\alpha$, $b = u/\alpha$, $c = \theta/\alpha$, and

$$F(\gamma, w) = \frac{1}{2\mathrm{i}}\left[\frac{e^{\gamma(w+\mathrm{i}A)}}{1 - e^{w+\mathrm{i}A}} - \frac{e^{\gamma(w-\mathrm{i}A)}}{1 - e^{w-\mathrm{i}A}}\right].$$

Splitting positive and negative frequencies gives

$$\Phi_{\beta} = F(\beta, -b + \mathrm{i}c) + F(1 - \beta, -b - \mathrm{i}c).$$

The denominators give the asserted joint local bound. For $0 < \beta < 1$, a decay rate is $\min(\beta, 1 - \beta)/\alpha$. At $\beta = 0$, the zero-order sine vanishes and rate $1/\alpha$ suffices.

Insert the integral representation

$$I_{\nu}(\zeta) = \frac{1}{\pi}\int_{0}^{\pi} e^{\zeta\cos\nu}\cos(\nu\nu)\,\mathrm{d}\nu - \frac{\sin(\pi\nu)}{\pi}\int_{0}^{\infty} e^{-\zeta\cosh u - \nu u}\,\mathrm{d}u$$

[24, Eq. 10.32.4]. Twisted Fourier summation of the first term gives the image indicator and its phases. An Abel factor $e^{-\varepsilon\nu_n}$ justifies summation first. The joint bound is locally integrable in (u, θ); it and the large-u decay justify the distributional limit. Combining fractions at $\beta = 0$ gives the last formula.

Proposition (Image–diffraction representation). *At generic pairs,*

$$G_{F,E} = \sum_{k \in \mathcal{K}(\theta)} \frac{e^{2\pi\mathrm{i}\beta k}}{4\pi^{2}(R_{k}^{2} + \Delta\tau^{2})} - \frac{1}{\pi\alpha}\int_{0}^{\infty} \frac{\Phi_{\beta}(u,\theta)}{4\pi^{2}(\rho(u)^{2} + \Delta\tau^{2})}\,\mathrm{d}u.$$

Across image boundaries the right-hand side denotes the total joint distribution fixed by the heat kernel.

Proof. Apply Lemma inside the four-dimensional heat integral and use

$$\int_{0}^{\infty} (4\pi s)^{-2} e^{-D^{2}/(4s)}\,\mathrm{d}s = \frac{1}{4\pi^{2}D^{2}}.$$

At generic pairs the integrals converge absolutely. The joint local bound gives the tested boundary prescription.

Theorem (Analytic direct subtraction at every regular point). *At every X_{0} with $r_{0} > 0$, choose a sufficiently small Euclidean developing neighbourhood V_{E} and parallel frame. Then*

$$H_{F,E}(X, X') = G_{F,E}(X, X') - \frac{1}{4\pi^{2}\sigma_{E}^{2}(X, X')}$$

is jointly real-analytic on $V_{E} \times V_{E}$, including its diagonal, Hermitian, and a bisolution of the flat Euclidean

Laplacian.

No condition such as $1/(2\alpha) \notin \mathbb{N}$ *is required.*

Proof. In developing Cartesian coordinates the two terms are fundamental distributions of the same local operator in each variable. Their delta sources cancel. The difference is therefore annihilated, distributionally, by Δ_X and $\Delta_{X'}$, hence by the elliptic operator $\Delta_X + \Delta_{X'}$ on the product. Elliptic regularity gives smoothness and analytic elliptic regularity gives joint real analyticity [25, 26]. Hermiticity follows from the self-adjoint heat kernel.

This treats the total distribution and never differentiates a moving image-sector indicator in isolation.

Near a generic diagonal pair, $H_{F,E}$ is precisely with the $k = 0$ term removed. At a boundary parameter its analytic representative is the one in Theorem .

For $\alpha = 1/N$ and $N\beta \in \mathbb{Z}$, every $\sin(\pi\nu_n)$ vanishes and the kernel is the full finite character-weighted quotient image kernel. For even N, the strict image set at zero angle omits the two boundary representatives of one quotient image. The full quotient sum or an equivalent limiting prescription must be used; simply dropping them is incorrect. For incompatible characters diffraction need not vanish.

Theorem (Local Lorentzian Hadamard form). *Every regular Lorentzian point has a sufficiently small developing neighbourhood on which*

$$W_F(X, X') = \frac{1}{4\pi^2[|\mathbf{X} - \mathbf{X}'|^2 - (t - t' - \mathrm{i}0)^2]} + H_F(X, X'),$$

where H_F *is smooth and locally analytic. It is obtained by continuing the time difference of* $H_{F,E}$.

Proof. For positive Euclidean separation the spectral kernel is $e^{-a\sqrt{B_F}}/(2\sqrt{B_F})$. Its boundary value at $a = \varepsilon + \mathrm{i}(t - t')$ is W_F, by the spectral theorem and Lemma . The analytic remainder has a holomorphic extension near the regular diagonal. After shrinking the neighbourhood it continues smoothly in the time difference. The direct term gives the displayed boundary value. The local geometry and connection are flat and the field is massless, so the local logarithmic Hadamard coefficient vanishes [33].

Remark (Local, not global, microlocal statement). A global Radzikowski characterization on this non-globally-hyperbolic host is not proved [34]. Extension-dependent geometric and diffractive fronts require a separate analysis [21, 42]. A broken path through the completed apex is not a regular null geodesic of M. No statement is made at $r = 0$.

Renormalization, complex first jets, and Ward identities

Assumption (Renormalization class). On regular developing neighbourhoods use local covariant Hadamard point splitting, the usual scaling and smooth mass-dependence conditions, and zero Minkowski-vacuum normalization. The flat connection is an external background.

No state-dependent or holonomy-dependent nonlocal subtraction, and no apex-supported counterterm, is included.

This is the local framework of [28, 31, 32, 33]. Its local formulas are applied on regular charts, not as a globally hyperbolic existence theorem for all of M.

Let brackets denote coincidence and let H be the regular complex two-point kernel in a parallel frame. Put

$$F = [H].$$

For one real field use $h = \mathrm{Re}\, H$, and for complex stress use $h = 2\mathrm{Re}\, H$. For a smooth symmetric real bisolution define

$$= (1 - 2\xi)[\nabla_{(\mu}\nabla_{\nu)'}h] + \left(2\xi - \frac{1}{2}\right)g_{\mu\nu}[\nabla_{\rho}\nabla^{\rho'}h]$$

$$-\xi[(\nabla_{\mu}\nabla_{\nu} + \nabla_{\mu'}\nabla_{\nu'})h].$$

Indices are parallel transported before coincidence.

For the charged field,

$$\langle j_{\mu}\rangle_{\mathrm{ren}} = \mathrm{i}e[(D_{\mu} - D^{*}_{\mu'})H], \qquad j_{\mu} = \mathrm{i}e[\Phi^{\dagger}D_{\mu}\Phi - (D_{\mu}\Phi)^{\dagger}\Phi].$$

The two slots carry opposite connection characters.

Lemma (Regular-part identities and ambiguities). *Under Assumption :*

1. *The Wick-square ambiguity $am^2 + bR$, the usual stress ambiguities, and local background-gauge-curvature terms vanish on the regular part. The possible current ambiguity proportional to $\nabla^{\nu}F_{\nu\mu}$ also vanishes.*

2. *Local conservation corrections and the regular conformal trace anomaly vanish.*

3. *For $F_h = [h]$,*

$$\nabla^{\mu}T_{\mu\nu} = 0, \qquad T^{\mu}{}_{\mu} = \frac{1}{2}(6\xi - 1)\mathrm{Box}_g F_h,$$

$$T_{\mu\nu}(\xi) - T_{\mu\nu}(1/6) = \left(\xi - \frac{1}{6}\right)(g_{\mu\nu}\mathrm{Box}_g - \nabla_{\mu}\nabla_{\nu})F_h.$$

For the complex field $F_h = 2F$.

4. *The current defined by is real and conserved. These statements do not classify completion-supported distributions or contact terms in higher products.*

Proof. All admissible local geometric ambiguity and anomaly expressions vanish when $m = 0$, the Riemann tensor is zero, and the external connection curvature is zero. Holonomy is not a local curvature invariant.

In a parallel Cartesian chart put

$$A_{\mu\nu} = [\partial_{\mu}\partial_{\nu'}h], \qquad K = A^{\mu}{}_{\mu}.$$

Symmetry, the bisolution equations, and Synge's rule give

$$\mathrm{Box}\,F_h = 2K, \qquad \partial_{\mu}\partial_{\nu}F_h = 2[\partial_{\mu}\partial_{\nu}h] + 2A_{\mu\nu}, \qquad \partial^{\mu}A_{\mu\nu} = \frac{1}{2}\partial_{\nu}K.$$

Thus

$$T_{\mu\nu} = A_{\mu\nu} - \xi\partial_{\mu}\partial_{\nu}F_h + \left(2\xi - \frac{1}{2}\right)g_{\mu\nu}K.$$

Taking a divergence, trace, and coupling difference proves (iii). Hermiticity makes the current real. Applying a divergence to its coincidence expression gives $ie[(\mathrm{Box}_x - \mathrm{Box}_{x'})H] = 0$.

Remark (The complex first-jet distinction). The symmetric real part suffices for the stress tensor but not for the current. If H is replaced by $\mathrm{Re}\,H$ before taking the antisymmetric first derivative, the azimuthal current is lost.

In the periodic gauge,

$$H_{\mathrm{per}} = e^{-\mathrm{i}\beta(\varphi - \varphi')} H_{\mathrm{tw}}.$$

The covariant derivatives in cancel this phase. The prescription is therefore gauge invariant. The parametrix must be transformed with the same parallel factor.

Remark (What ``no subtraction needed'' can mean). For the azimuthal current computed by time splitting at $\Delta\varphi = 0$, the direct parametrix has zero azimuthal derivative difference. No extra subtraction appears in that particular calculation.

It is false that the derivative difference of $(4\pi^2\sigma_E^2)^{-1}$ vanishes along every splitting. For example a spatial derivative of that singular function is generally nonzero and divergent. Equation , using the regular kernel, is the splitting-independent definition.

Exact Friedrichs observables and analytical checks

Define

$$P_\beta(\alpha) = \alpha^{-2}[1 - 6\beta(1-\beta)] - 1,$$

$$C_\beta(\alpha) = \alpha^{-4}[1 - 30\beta^2(1-\beta)^2] - 1.$$

Lemma (Time-split expansion). *For $r = r'$, $\Delta\varphi = \Delta z = 0$, and $\Delta\tau = s$,*

$$H_{F,E}(s) = \frac{P_\beta}{48\pi^2 r^2} - \frac{C_\beta + 10P_\beta}{2880\pi^2 r^4} s^2 + O(s^4).$$

Proof. The angular sum is

$$S_\beta(a) = \frac{\cosh((1/2-\beta)a)}{\sinh(a/2)} = \frac{2}{a} + B_2(\beta)a + \frac{B_4(\beta)}{12}a^3 + O(a^5),$$

where

$$B_2(\beta) = \beta^2 - \beta + \frac{1}{6}, \qquad B_4(\beta) = \beta^4 - 2\beta^3 + \beta^2 - \frac{1}{30}.$$

Also

$$\eta = 2\operatorname{arsinh}\frac{s}{2r}, \qquad \eta^{-2} = \frac{r^2}{s^2} + \frac{1}{12} - \frac{s^2}{240r^2} + O(s^4),$$

and

$$\frac{S_\beta(\eta/\alpha)}{\alpha \sinh \eta} = \frac{2}{\eta^2} + \frac{B_2}{\alpha^2} - \frac{1}{3} + \eta^2\left[\frac{B_4}{12\alpha^4} - \frac{B_2}{6\alpha^2} + \frac{7}{180}\right] + O(\eta^4).$$

Multiply by $(8\pi^2 r^2)^{-1}$, subtract $(4\pi^2 s^2)^{-1}$, and use $P_\beta = 6B_2/\alpha^2 - 1$, $C_\beta = -30B_4/\alpha^4 - 1$. Theorem identifies these with the unique regular coincidence limits.

Theorem (Friedrichs Wick squares and stress tensors). *For one neutral real scalar,*

$$\langle \varphi^2 \rangle_{\mathrm{ren}} = \frac{\alpha^{-2} - 1}{48\pi^2 r^2},$$

$$\langle T^\mu{}_\nu \rangle_{\mathrm{ren}} = \frac{\alpha^{-4} - 1}{1440\pi^2 r^4}\operatorname{diag}(1, 1, -3, 1)$$

$$+\left(\xi - \frac{1}{6}\right)\frac{\alpha^{-2} - 1}{48\pi^2 r^4}\operatorname{diag}(4, -2, 6, 4).$$

For one charged complex scalar,

$$\langle \Phi^\dagger \Phi \rangle_{\mathrm{ren}} = \frac{P_\beta}{48\pi^2 r^2},$$

$$\langle T^\mu{}_\nu \rangle_{\mathrm{ren}} = \frac{C_\beta}{720\pi^2 r^4}\operatorname{diag}(1, 1, -3, 1)$$

$$+2\left(\xi - \frac{1}{6}\right)\frac{P_\beta}{48\pi^2 r^4}\operatorname{diag}(4, -2, 6, 4).$$

Their traces are respectively

$$\left(\xi - \frac{1}{6}\right)\frac{\alpha^{-2} - 1}{4\pi^2 r^4}, \qquad \left(\xi - \frac{1}{6}\right)\frac{P_\beta}{2\pi^2 r^4}.$$

Proof. The Wick squares follow from Lemma . For stress first apply the real differential expression to $\operatorname{Re} H_{F,E}$.

Reflections, angular evenness of the real part, translations, boosts, and scaling imply

$$T^{\mu}{}_{\nu} = r^{-4}\operatorname{diag}(a, b, c, a).$$

Radial conservation gives $c = -3b$.

For $F = P_{\beta}/(48\pi^2 r^2)$,

$$[\partial_{\rho}\partial^{\rho'} H_{F,E}] = \frac{1}{2}\Delta_E F = \frac{P_{\beta}}{24\pi^2 r^4}.$$

The time-split identities give

$$T^{\tau}{}_{\tau} = -H''_{F,E}(0) + \left(2\xi - \frac{1}{2}\right)\frac{P_{\beta}}{24\pi^2 r^4} = \frac{C_{\beta}}{1440\pi^2 r^4} + \left(\xi - \frac{1}{6}\right)\frac{4P_{\beta}}{48\pi^2 r^4}.$$

At conformal coupling the trace vanishes, hence $a = b$. For other couplings use

$$(g\mathrm{Box} - \nabla\nabla)^{\mu}{}_{\nu} r^{-2} = r^{-4}\operatorname{diag}(4, -2, 6, 4).$$

Take $\beta = 0$ for one real field; multiply by two for one complex stress tensor. The trace formulas follow directly.

At conformal coupling the neutral energy density is negative:

$$\varepsilon = -\frac{\alpha^{-4} - 1}{1440\pi^2 r^4}.$$

At $\alpha = 1/2$, the single nontrivial image gives

$$H_E(s) = \frac{1}{4\pi^2(4r^2 + s^2)}.$$

Then

$$F = \frac{1}{16\pi^2 r^2}, \qquad H''_E(0) = -\frac{1}{32\pi^2 r^4}, \qquad \frac{1}{2}\Delta_E F = \frac{1}{8\pi^2 r^4},$$

so $T^{t}{}_{t} = 1/(96\pi^2 r^4)$. This is an independent sign check.

Theorem (Friedrichs induced current). *For one charged complex field,*

$$\langle j_\varphi \rangle_{\mathrm{ren}} = -\frac{e\,\beta(1-\beta)(1-2\beta)}{12\pi^2\alpha^2 r^2},$$

$$\langle j_\varphi^{\wedge} \rangle_{\mathrm{ren}} = -\frac{e\,\beta(1-\beta)(1-2\beta)}{12\pi^2\alpha^3 r^3}, \qquad j_t = j_r = j_z = 0.$$

It is conserved and independent of ξ. With the fractional-part convention it is periodic in flux, odd under $\beta \mapsto 1-\beta$*, and zero at* $0, 1/2$.

Proof. Time split at $\Delta\varphi = 0$. The direct azimuthal term vanishes. Differentiating ,

$$\partial_{\Delta\varphi} G_{F,E} = \frac{\mathrm{i}\Sigma(a)}{8\pi^2 \alpha r^2 \sinh\eta},$$

where

$$\Sigma(a) = \sum_n q_n e^{-|q_n|a} = -\frac{\mathrm{d}}{\mathrm{d}a}\frac{\sinh((1/2-\beta)a)}{\sinh(a/2)} = \frac{\beta(1-\beta)(1-2\beta)}{3}\,a + O(a^3).$$

Since $\partial_\varphi - \partial_{\varphi'} = 2\partial_{\Delta\varphi}$, gives the result. The other components vanish by the time and longitudinal evenness and radial endpoint symmetry. Conservation follows from $j^r = 0$ and angular independence. The remaining properties follow from the polynomial and mode relabeling.

Diffraction identity and quotient sums

Corollary (Untwisted diffraction identity). *If* $1/(2\alpha) \notin \mathbb{N}$, *then*

$$\sum_{0<|k|<1/(2\alpha)} \csc^2(\pi\alpha k) - \frac{1}{\pi\alpha}\int_0^\infty \frac{\sin(\pi/\alpha)\,\mathrm{d}u}{[\cosh(u/\alpha)-\cos(\pi/\alpha)]\cosh^2(u/2)} = \frac{\alpha^{-2}-1}{3}.$$

For $\alpha > 1/2$, *the image sum is empty, and on a sufficiently small diagonal neighbourhood the regular remainder is wholly represented by diffraction.*

Proof. At coincidence $R_k^2 = 4r^2\sin^2(\pi\alpha k)$ and $\rho(u)^2 = 4r^2\cosh^2(u/2)$. Multiply the subtracted contour representation by $16\pi^2 r^2$.

This representation does not make vacuum polarization a retarded signal emitted from an apex event.

Proposition (Exact quotient identities). *For $N \geq 2$,*

$$\sum_{k=1}^{N-1} \csc^2 \frac{\pi k}{N} = \frac{N^2-1}{3}, \qquad \sum_{k=1}^{N-1} \csc^4 \frac{\pi k}{N} = \frac{(N^2-1)(N^2+11)}{45}.$$

For $j = 0, \ldots, N-1$, put $u = j(N-j)$. Then

$$\sum_{k=1}^{N-1} e^{2\pi i j k/N} \csc^2 \frac{\pi k}{N} = \frac{N^2-1}{3} - 2u,$$

$$\sum_{k=1}^{N-1} e^{2\pi i j k/N} \csc^4 \frac{\pi k}{N} = \frac{(N^2-1)(N^2+11)}{45} - \frac{2}{3}u(u+2).$$

Proof. Logarithmic differentiation of the sine product gives

$$\sum_{k=0}^{N-1} \csc^2\left(x + \frac{\pi k}{N}\right) = N^2 \csc^2(Nx).$$

Compare constant and quadratic terms, using $\frac{1}{2}(\csc^2 x)'' = 3\csc^4 x - 2\csc^2 x$. This proves the untwisted identities.

For the twisted sums, the discrete second difference of the first sum is 4 away from $j = 0$ modulo N, and that of the second is minus four times the first. The displayed polynomials satisfy these equations and the periodic endpoint values; the periodic homogeneous second-difference equation has only constants. The known $j = 0$ values fix those constants.

At $\alpha = 1/N$, $\beta = j/N$, the full quotient kernel and these identities reproduce the Wick square and time-split stress coefficients exactly. This is an all-N analytical check, distinct from the reported finite symbolic runs.

Flat and flux limits

At fixed $r > 0$, for $\beta = 0$,

$$P_0 = \frac{\mu}{\pi} + O(\mu^2), \qquad C_0 = \frac{2\mu}{\pi} + O(\mu^2),$$

hence

$$\frac{\langle \varphi^2 \rangle_{\mathrm{ren}}}{\mu} \to \frac{1}{48\pi^3 r^2}, \qquad \frac{\langle T^t{}_t \rangle_{\mathrm{ren}}}{\mu} \to \frac{1}{720\pi^3 r^4} \quad (\xi = 1/6).$$

These are not uniform apex or wave-front limits.

At fixed nonzero β, $\alpha \to 1$ retains the punctured flat-space Aharonov–Bohm effect:

$$F \to -\frac{\beta(1-\beta)}{8\pi^2 r^2},$$

and the complex conformal tensor coefficient tends to $-\beta^2(1-\beta)^2/(24\pi^2 r^4)$. The conformal energy may have either sign as the parameters vary.

Boundary data and the complete transverse product ledger

The radial expression in angular mode n is

$$-\partial_r^2 - r^{-1}\partial_r + \nu_n^2 r^{-2} \quad \text{in} L^2((0,\infty), r\,\mathrm{d}r).$$

The deficient modes are

$$D_\beta = \{n : |n+\beta| < \alpha\}.$$

For $0 < \beta < 1$, these can only be $0, -1$:

$$0 \in D_\beta \Leftrightarrow \beta < \alpha, \qquad -1 \in D_\beta \Leftrightarrow 1-\beta < \alpha.$$

There are two precisely when $1-\alpha < \beta < \alpha$. If there are no deficient modes, Friedrichs is the unique transverse extension.

For $0 < \nu_j < 1$, boundary data are

$$u_j(r) = a_j r^{-\nu_j} + b_j r^{\nu_j} + o(r^{\nu_j}).$$

The integrated Green form is, with one fixed overall sign convention,

$$2\sum_j \nu_j(\bar{a}_j b_j' - \bar{b}_j a_j').$$

Its null Lagrangians classify the extensions. Write $N = \mathrm{diag}\,(\nu_j)$.

Definition (Boundary relation, including boundary charts). Choose a subspace $W \subseteq \mathbb{C}^d$, an isometric inclusion

$V : \mathbb{C}^{\dim W} \to \mathbb{C}^d$ with range W, and a Hermitian operator H_W on $\mathbb{C}^{\dim W}$. The boundary condition is

$$a \in W, \qquad V^\dagger N b = H_W V^\dagger a.$$

The component of Nb in $W^\perp$ is unrestricted.

For $W = \mathbb{C}^d$, this is

$$b = N^{-1} H a, \qquad H = H^\dagger.$$

For $W = \{0\}$, it is Friedrichs. The diagonal choice $H_{jj} = v_j \lambda_j$ gives $b_j = \lambda_j a_j$. The cases $\lambda_j = \infty$ are represented by removing that direction from W, not by an ordinary finite matrix entry.

Every Lagrangian relation has this form. Indeed, let W be its projection onto the a-space. The vertical part is $W^\perp$ in the Nb-variable, and the remaining graph is Hermitian on W. This is the finite-dimensional relation version of the usual $U(d)$ parametrization.

For $\beta = 0$, only $n = 0$ is deficient and

$$u_0(r) = a \log r + b + o(1).$$

Its extensions are Friedrichs or

$$b = -a \log L, \qquad 0 < L < \infty, \qquad u_0 \sim a \log (r/L),$$

denoted L_L.

Krein formula, infrared control, and nonnegative states

For $0 < \nu < 1$, set

$$s_\nu = \frac{\pi}{2 \sin \pi\nu}, \qquad e_\nu(p) = s_\nu \frac{(p/2)^{-\nu}}{\Gamma(1-\nu)}, \qquad d_\nu(p) = \frac{(p/2)^{\nu}}{\Gamma(1+\nu)},$$

$$\Delta_\nu(p) = \nu \frac{\Gamma(1-\nu)}{\Gamma(1+\nu)} (p/2)^{2\nu}, \qquad c_0(\nu) = \frac{2 \sin \pi\nu}{\pi}.$$

Let E, D, Δ, S be the corresponding diagonal matrices, with $S_{jj} = s_{\nu_j}$. In particular $d_\nu e_\nu = 1/(2\nu)$.

Theorem (Resolvent correction on every boundary chart). *For the extension , define*

$$C_W(p) = \frac{1}{2} E^{-1} V [H_W + V^\dagger \Delta(p) V]^{-1} V^\dagger E^{-1}.$$

At values where the inverse exists, the transverse resolvent difference is

$$(B_{\perp,W} + p^2)^{-1}(x, x') - (B_{\perp,F} + p^2)^{-1}(x, x')$$

$$= \sum_{jk} \psi_j(\varphi) K_{\nu_j}(pr) C_{W,jk}(p) K_{\nu_k}(pr') \overline{\psi_k(\varphi')}.$$

In the full finite chart,

$$C(p) = \frac{1}{2} E^{-1} (H + \Delta)^{-1} E^{-1} = (2EHE + S)^{-1}.$$

For a diagonal mode,

$$c_\lambda(p) = \frac{c_0(\nu)}{1 + \lambda \frac{\Gamma(1+\nu)}{\Gamma(1-\nu)} (p/2)^{-2\nu}}.$$

For E_0, $c = c_0(\nu)$. *For* L_L,

$$c_L(p) = \frac{1}{\log(p/\kappa_L)}, \qquad \kappa_L = \frac{2e^{-\gamma_E}}{L}.$$

Proof. The resolvent difference is a finite sum of decaying homogeneous solutions $K_{\nu_j}(pr)$ in each slot. Near zero,

$$K_\nu(pr) = e_\nu(p) r^{-\nu} - s_\nu d_\nu(p) r^\nu + o(r^\nu).$$

For a source column w, the Friedrichs term contributes Dw to the regular boundary coefficient. Thus

$$a = ECw, \qquad b = Dw - DSCw.$$

Set $a = V\zeta$. Since $ND = \frac{1}{2}E^{-1}$ and $NDSE^{-1} = \Delta$, condition becomes

$$[H_W + V^\dagger \Delta V]\zeta = \frac{1}{2} V^\dagger E^{-1} w.$$

Then $Cw = E^{-1}V\zeta$, proving . For the full chart use $2E\Delta E = S$. The diagonal case follows immediately.

For $v = 0$, the expansion $K_0(pr) = -\log(pr/2) - \gamma_E + o(1)$ gives $c_L^{-1} = \log(pL/2) + \gamma_E$.

Lemma (Uniform positive Krein bound). *If $H_W \geq 0$, then*

$$0 \leq C_W(p) \leq S^{-1}, \qquad p > 0,$$

as Hermitian matrices.

Proof. The inverse in is bounded above by $(V^\dagger \Delta V)^{-1}$. Moreover

$$V(V^\dagger \Delta V)^{-1} V^\dagger \leq \Delta^{-1},$$

because conjugation by $\Delta^{1/2}$ turns the left side into the orthogonal projection onto $\operatorname{ran}(\Delta^{1/2}V)$. Consequently

$$C_W \leq \frac{1}{2} E^{-1}\Delta^{-1}E^{-1} = S^{-1}.$$

Theorem (Euclidean correction and spectral-state existence). *If $H_W \geq 0$, the massless spectral covariance is finite and continuous on compact tests. Its Euclidean kernel is $G_{W,E} = G_{F,E} + \delta G_W$, where*

$$\delta G_W(X, X') = \frac{1}{4\pi^2\alpha} \int_0^\infty p \, \mathrm{d}p \, J_0\left(p\sqrt{\Delta\tau^2 + \Delta z^2}\right) \sum_{jk} e^{\mathrm{i}q_j\varphi} K_{\nu_j}(pr) \times C_{W,jk}(p) K_{\nu_k}(pr') e^{-\mathrm{i}q_k\varphi'}.$$

This correction is jointly real-analytic at every pair of regular Euclidean points. The spectral state has the local Hadamard form of Theorem .

The state remains t, z-translation invariant and (t, z)-boost invariant. Rotation and dilation invariance hold exactly when the extension has those symmetries.

Proof. Fourier transform the product resolvent in the two Euclidean variables (τ, z). The radial angular integral in their momentum plane gives J_0, and the normalized transverse modes give the factor $(4\pi^2\alpha)^{-1}$.

At small p, Lemma and $K_{\nu_j}(pr) = O(p^{-\nu_j})$, locally uniformly in $r > 0$, bound every term by

$$C_K p^{1-\nu_j-\nu_k}.$$

This is integrable because $\nu_j, \nu_k < 1$. Spatial and angular derivatives preserve this integrability; time and longitudinal derivatives improve it. At large p, the bound and the K-Bessel asymptotic give exponential decay $e^{-p(r+r')}$ times a polynomial.

In particular the equal-time correction is bounded on fixed compact regular spatial pairs. The quadratic form of $B^{-1/2}$, first defined with a positive spectral or mass cutoff, is the Friedrichs form plus this finite correction. Monotone convergence in the spectral integral removes the cutoff. Lemma therefore supplies the remaining Friedrichs part. This proves compact-test finiteness and continuity rather than inferring them from $B \geq 0$.

For analyticity, use the two-dimensional Fourier representation before angular integration. On a small complex neighbourhood of any real regular pair, the imaginary displacement in (τ, z) is less than the positive real part of $r + r'$. The exponential large-p estimate then persists. The small-p bound is locally uniform in that neighbourhood. Holomorphic parameter integration gives a holomorphic extension, hence real analyticity.

The difference is a Euclidean bisolution. Its continuation is smooth locally, so the same direct Hadamard singularity remains. The spectral construction gives positivity, and the product mass-shell argument proves boosts.

Remark (No universal locality assertion for arbitrary relations). The preceding theorem constructs the extension-selected spectral state and its local near-diagonal Hadamard form. It does not automatically extend the Friedrichs intrinsic finite-propagation theorem to every rotation-breaking boundary relation. The commutator is always the one belonging to the chosen extension.

Spectrum, positivity, and scale

Theorem (Complete product positivity ledger). *For* $0 < \beta < 1$*:*

1. *The negative transverse eigenvalues are* $-\kappa^2$ *satisfying*

$$\det\left[H_W + V^\dagger \Delta(\kappa) V\right] = 0.$$

Their number, with multiplicity, is the number of negative eigenvalues of H_W. There are no transverse zero eigenfunctions. Thus the full product operator is nonnegative exactly when $H_W \geq 0$.

2. *In a full finite chart, rotation invariance is equivalent to diagonality of H. Rotation by γ sends $H_{0,-1}$ to $e^{i\gamma}H_{0,-1}$. On boundary charts the corresponding condition is invariance of the whole relation.*

3. *If the deficient orders are distinct, the dilation-invariant relations are exactly the 2^d coordinate choices of Friedrichs or E_0. If $\beta = 1/2$ and $\alpha > 1/2$, there is in addition*

$$a \in \mathbb{C}v, \qquad b \perp v, \qquad [v] \in \mathbb{CP}^1.$$

This is E_0 on v and Friedrichs on $v^\perp$. For $\beta = 0$, every L_L has one transverse eigenvalue $-\kappa_L^2$. Friedrichs is the only nonnegative massless extension in the untwisted product class.

Proof. A negative transverse eigenfunction is a linear combination of $K_{\nu_j}(\kappa r)\psi_j$. Its boundary condition is precisely the displayed determinant condition, by the calculation in Theorem .

The compressed matrix $V^\dagger \Delta(\kappa) V$ increases strictly in Loewner order from zero to infinity. The ordered eigenvalues of its sum with H_W therefore increase strictly and continuously from the eigenvalues of H_W to positive infinity. Exactly its initially negative eigenvalues cross zero, counted with multiplicity. The resolvent difference has finite rank, so the transverse essential spectrum remains $[0, \infty)$. Zero-energy radial solutions $r^{\pm\nu}$, or $1, \log r$, cannot be square integrable at infinity.

Rotations multiply each boundary component by $e^{iq_j\gamma}$. For two deficient modes their charge difference is one, giving the stated transformation.

Modulo a common scalar normalization, dilations act by $a_j \mapsto s^{\nu_j} a_j$, $b_j \mapsto s^{-\nu_j} b_j$. With distinct positive orders all weights are distinct, so an invariant Lagrangian is a coordinate sum containing exactly one of each paired a_j, b_j axes. At equal orders an invariant Lagrangian is $W_a \oplus W_a^\perp$ in the a, b weight spaces. The intermediate one-dimensional choices give $\mathbb{CP}^1$.

Finally $K_0(\kappa r)$ matches $\log(r/L)$ at $\kappa = 2e^{-\gamma_E}/L$.

A transverse eigenvalue does not remain an $L^2(N)$ eigenvalue after adding the noncompact z-direction. It produces the branch

$$p_z^2 - \kappa^2,$$

a continuous band containing negative spectrum. This distinction is essential in the instability argument.

In a finite chart, H_{jk} has length dimension $\text{length}^{-\nu_j - \nu_k}$. Thus $H_{0,-1}$ has dimension $\text{length}^{-1/\alpha}$. Generic finite off-diagonal data introduce a length. The half-holonomy scale-invariant line family is a boundary relation and is not an exception represented by a finite nonzero off-diagonal H.

Unstable extensions and the two projected alternatives

Theorem (No stationary finite-second-moment local state). *Suppose a product extension has a negative transverse eigenvalue $-\kappa^2$. There is no time-translation-invariant state on its extension-selected compact-test field algebra having finite second moments for every compact smooth field test.*

In particular, no regular spectral ground state or regular KMS state with that finite-moment property exists. This excludes every massless L_L*, every diagonal combination with some* $\lambda_j < 0$*, and every boundary relation with* $H_W \not\geq 0$.

Proof. The full spatial operator is bounded below. Choose a compact smooth spatial section v with nonzero spectral measure on some interval $[-b, -a] \subset (-\infty, 0)$. Such a section exists: compact tests are dense, and the negative spectral projection is nonzero. For a charged field use the underlying real phase space.

Choose a nonnegative, nonzero $\chi \in C_c^\infty(\mathbb{R})$, supported in $[-A, A]$, and let $f(t,x) = \chi(t)v(x)$. The commutator pairing of f and its time translate f_T is obtained by integrating

$$E(T + s - s') = \frac{\sin((T + s - s')\sqrt{B})}{\sqrt{B}}$$

against $\chi(s)\chi(s')$.

For $\lambda = -\omega^2 < 0$, this contribution equals

$$\frac{\sinh(\omega T)}{\omega}\Big(\int \chi(s)e^{\omega s}\,ds\Big)\Big(\int \chi(s)e^{-\omega s}\,ds\Big).$$

It is positive and grows at least as a positive constant times $e^{\sqrt{a}\,T}$ on a smaller negative spectral interval. For nonnegative spectrum, the elementary bound $|\sin(u\sqrt{\lambda})/\sqrt{\lambda}| \le |u|$ bounds the entire contribution by $C(T + 2A)\|v\|^2$. The commutator magnitude is therefore unbounded in T.

In an invariant state with finite $\omega(\Phi(f)^2)$, Cauchy–Schwarz gives

$$|\omega([\Phi(f_T), \Phi(f)])| \le 2\,\omega(\Phi(f)^2),$$

independently of T, a contradiction.

Remark (The unstable-data proof retained). The SAE source also gives a phase-space proof. In a transverse bound channel choose longitudinal data supported in $|p_z| \le \kappa/2$, and put $\Omega = \sqrt{\kappa^2 - p_z^2}$. For $u_\pm = (f, \pm\Omega f)$,

$$|\sigma(T_t u_+, u_-)| = 2\langle f, \Omega e^{t\Omega} f\rangle \ge 2e^{\sqrt{3}\kappa t/2}\langle f, \Omega f\rangle.$$

Robertson's inequality contradicts invariance if both variances are finite. That argument concerns an algebra containing these generally noncompact data. Theorem avoids silently identifying that data algebra with the compact-test algebra.

Principal-value reference

For an unstable extension let G_{PV} denote the Euclidean resolvent integral with principal values at its negative transverse poles. It is a useful reference distribution, not a positive spectral ground-state covariance. The finite-rank correction to the Friedrichs kernel is locally analytic at regular diagonal pairs, because its high-momentum part is exponentially decaying and its finite poles are treated as principal-value distributions.

Proposition (Projected constructions and their defects). *Suppose for simplicity there is one normalized*

transverse bound eigenfunction $b(x_\perp)$ at $-\kappa^2$.

1. *Removing the entire transverse bound channel gives*

$$G_{\mathrm{cont}} = G_{\mathrm{PV}} - b(x_\perp)\bar{b}(x'_\perp)Q_\kappa(\Delta),$$

where

$$Q_\kappa(\Delta) = \mathrm{PV}\int \frac{\mathrm{d}^2k}{(2\pi)^2}\frac{e^{\mathrm{i}k\cdot\Delta}}{k^2-\kappa^2} = -\frac{1}{4}Y_0(\kappa|\Delta|), \qquad \Delta = (\Delta\tau, \Delta z).$$

Since $Q_\kappa \sim (2\pi)^{-1}\log(1/|\Delta|)$, the subtracted Wick-square limit is $-\infty$ where $b \neq 0$. The kernel also has extra singularities at $\Delta\tau = \Delta z = 0$ for distinct transverse points. It is not locally Hadamard for the original field.

2. *Restricting only to the full nonnegative spectral subspace keeps the bound-channel modes with $|p_z| > \kappa$. Its Euclidean kernel is*

$$G_+ = G_{\mathrm{PV}} + b\bar{b}' D(\Delta\tau, \Delta z),$$

where

$$D(0, z) = 0, \qquad D(\tau, 0) = \frac{1}{4}\mathbf{H}_0(\kappa|\tau|).$$

The correction tends to zero at the diagonal, so its Wick-square value agrees with the principal-value reference. It is not a smooth regular remainder:

$$D(\tau, 0) = \frac{\kappa|\tau|}{2\pi} + O(|\tau|^3).$$

Thus the usual covariant stress point-splitting prescription does not apply. The projected covariance is not boost invariant. Both constructions have projected equal-time CCR and are nonlocal modifications of the original field algebra.

Proof. Spectrally,

$$R_\perp(-p^2) = R_{\mathrm{cont}}(-p^2) + \frac{P_b}{p^2-\kappa^2}.$$

Fourier transformation in (τ, z) proves (a). The p^{-2} bound-channel tail is precisely the logarithm removed from the full reference. The continuum resolvent difference therefore has the compensating algebraic tail; it must not be described as exponentially decaying.

For (b), the retained bound contribution is

$$\int_{|p_z|>\kappa} \frac{\mathrm{d}p_z}{2\pi} e^{\mathrm{i}p_z z} \frac{e^{-|\tau|\sqrt{p_z^2-\kappa^2}}}{2\sqrt{p_z^2-\kappa^2}}.$$

At $\tau = 0$ it is $-Y_0(\kappa|z|)/4$. At $z = 0$, substituting $u = \sqrt{p_z^2 - \kappa^2}$ gives

$$\frac{1}{2\pi}\int_0^\infty \frac{e^{-|\tau|u}}{\sqrt{u^2+\kappa^2}}\,\mathrm{d}u = \frac{1}{4}[\mathbf{H}_0 - Y_0](\kappa|\tau|).$$

Subtract Q_κ. The omitted compact momentum interval gives a bounded Fourier contribution tending to zero with τ, so the coincidence agreement is not restricted to the two axes. The two axis formulas prove failure of Euclidean rotational, hence Lorentzian boost, invariance. Projection replaces the spatial identity in the canonical commutator by the corresponding spectral projection.

These special-function identities can also be obtained by Laplace and Hankel transformation [24, 23]. The construction does not repair the original unstable dynamics.

Proposition (Logarithmic principal-value Wick square). *For* L_L,

$$\begin{aligned}\delta F_{\mathrm{PV}} &= \frac{1}{4\pi^2\alpha r^2}\mathrm{PV}\int_0^\infty \frac{xK_0(x)^2}{\log(x/\kappa_L r)}\,\mathrm{d}x \\ &= -\frac{1}{8\pi^2\alpha r^2 \ell}\left[1 + \frac{\log 2 - 1 - \gamma_E}{\ell} + O(\ell^{-2})\right], \qquad \ell = \log(\kappa_L r),\end{aligned}$$

as $\ell \to +\infty$ *or* $\ell \to -\infty$.

Proof. Set $y = \log x$ and $w(y) = e^{2y}K_0(e^y)^2$. This is a Schwartz function on $\mathbb{R}$. The principal-value integral is $\mathrm{PV}\int w(y)/(y-\ell)\,\mathrm{d}y$. Split into $|y| < |\ell|/2$, a neighbourhood of the moving pole, and the remaining tails. On the first region expand the reciprocal. Near the pole subtract $w(\ell)$ symmetrically and use Schwartz bounds on w'; all tail errors are smaller than any prescribed inverse power after enough moments are kept. Hence

$$\mathrm{PV}\int \frac{w(y)}{y-\ell}\,\mathrm{d}y = -\frac{1}{\ell}\int w - \frac{1}{\ell^2}\int yw + O(|\ell|^{-3}).$$

The Mellin formula in Appendix gives

$$\int_0^\infty xK_0^2\,\mathrm{d}x = \frac{1}{2}, \qquad \int_0^\infty xK_0^2 \log x\,\mathrm{d}x = \frac{1}{2}(\log 2 - 1 - \gamma_E).$$

Scale-invariant irregular extensions

Theorem (Single-mode corrections and shifted branches). *Let n be deficient with* $q = n + \beta \neq 0$ *and* $v = |q|/\alpha < 1$. *Take* E_0 *in that mode and Friedrichs in the others. Then*

$$\delta G_n = \frac{\sinh(v\eta)}{4\pi^2 \alpha r r' \sinh\eta} e^{\mathrm{i}q(\varphi-\varphi')}.$$

For one complex field,

$$\delta F = \frac{|q|}{4\pi^2\alpha^2 r^2}, \qquad \delta j_\varphi = -\frac{e\,q|q|}{2\pi^2\alpha^2 r^2}, \qquad \delta j_t = \delta j_r = \delta j_z = 0,$$

$$\delta T^t{}_t = \delta T^z{}_z = \frac{|q|(12\alpha^2\xi - 2\alpha^2 - q^2)}{6\pi^2\alpha^4 r^4},$$

$$\delta T^r{}_r = -\frac{|q|(6\alpha^2\xi - \alpha^2 + q^2)}{6\pi^2\alpha^4 r^4}, \qquad \delta T^\varphi{}_\varphi = -3\delta T^r{}_r.$$

Let $P_X(\beta)$ *be any Friedrichs polynomial formula for* $X = F, j_\varphi, T^\mu{}_\nu$, *regarded as a polynomial rather than periodically reduced. Then*

$$X_{\mathrm{E}_0^{(0)}} = P_X(\beta+1), \qquad X_{\mathrm{E}_0^{(-1)}} = P_X(\beta-1).$$

If both modes are deficient and both are irregular,

$$X_{\mathrm{E}_0^{(0)}\mathrm{E}_0^{(-1)}} = P_X(\beta+1) + P_X(\beta-1) - P_X(\beta).$$

Proof. For $c = c_0(v)$, the radial heat kernel is obtained by replacing I_ν by $I_{-\nu}$. The Laplace transform used in Proposition remains valid because $-v > -1$. Their difference gives .

Let $c = \cosh\eta$. Near $c = 1$,

$$\frac{\sinh(v\eta)}{\sinh\eta} = v + \frac{v(v^2-1)}{3}(c-1) + O((c-1)^2).$$

This proves regularity directly and supplies its second jet. Coincidence gives $\delta F = v/(4\pi^2\alpha r^2)$; the angular derivative gives $\delta j_\varphi = -2eq\,\delta F$. The time-split second derivative, conservation, and the improvement identity give the displayed stress components.

For the branch identity set

$$x = e^{-\eta/\alpha + i\Delta\varphi}, \qquad y = e^{-\eta/\alpha - i\Delta\varphi},$$

and fix their displayed logarithms. Define the entire function of the branch parameter

$$S(b) = \frac{x^b}{1-x} + \frac{y^{1-b}}{1-y}.$$

On $0 < b < 1$, this is the physical Friedrichs angular sum. Irregularizing mode 0 replaces x^β by $y^{-\beta}$; the result is $S(\beta+1)$. Irregularizing mode -1 similarly gives $S(\beta-1)$. The coincidence jets of this entire branch give the stated polynomial identities. Independent mode corrections add.

Remark (What is, and is not, entire). The physical function $\sum_n e^{-|n+\beta|a + i(n+\beta)b}$, extended by relabeling, is periodic in the flux parameter and is not the same entire branch for all complex β. The shift theorem uses S, with a fixed splitting of positive and negative frequencies. No physical equivalence between a changed extension and a mere integer gauge transformation is asserted.

For example, mode 0 gives

$$F_{\mathrm{E}_0^{(0)}} = \frac{\alpha^{-2}[1+6\beta(1+\beta)]-1}{48\pi^2 r^2},$$

$$j_{\varphi,\mathrm{E}_0^{(0)}} = -\frac{e\beta(1+\beta)(1+2\beta)}{12\pi^2\alpha^2 r^2},$$

and with

$$P = \alpha^{-2}[1+6\beta(1+\beta)]-1, \qquad C = \alpha^{-4}[1-30\beta^2(1+\beta)^2]-1.$$

At $\xi = 1/6$,

$$\delta T^{\mu}{}_{\nu} = -\frac{|q|^3}{6\pi^2\alpha^4 r^4}\mathrm{diag}\,(1, 1, -3, 1).$$

As $\beta \to 0^{+}$, the mode-0 correction vanishes. Under charge conjugation the mode-0 choice at β is exchanged with mode -1 at $1-\beta$, and the current changes sign.

The half-holonomy line family

Theorem ($\mathbb{CP}^1$ family). *Let $\beta = 1/2$, $\alpha > 1/2$, and $v = 1/(2\alpha)$. For*

$$v = (\cos\theta, \sin\theta\, e^{i\chi}),$$

take E_0 on v and Friedrichs on $v^{\perp}$. Then $C = c_0 v v^{\dagger}$, and

$$\delta G = \frac{\sinh(\eta/(2\alpha))}{4\pi^2\alpha r r' \sinh\eta} w(\varphi)\overline{w(\varphi')},$$

$$w(\varphi) = \cos\theta\, e^{i\varphi/2} + \sin\theta\, e^{i\chi} e^{-i\varphi/2}.$$

Put

$$u = \sin 2\theta\cos(\varphi - \chi), \qquad \pi = \sin 2\theta\sin(\varphi - \chi), \qquad k_{\xi} = 6\xi - 1.$$

For one complex field,

$$\delta F = \frac{1+u}{8\pi^2\alpha^2 r^2}, \qquad \delta j_{\varphi} = -\frac{e\cos 2\theta}{8\pi^2\alpha^2 r^2}, \qquad \delta j_r = \delta j_t = \delta j_z = 0,$$

$$\delta T^{t}{}_{t} = \delta T^{z}{}_{z} = \frac{8\alpha^2 k_{\xi}(1+u) - 2k_{\xi}u - 1}{48\pi^2\alpha^4 r^4},$$

$$\delta T^{r}{}_{r} = \frac{-4\alpha^2 k_{\xi}(1+u) - 2k_{\xi}u - 1}{48\pi^2\alpha^4 r^4},$$

$$\delta T^{\varphi}{}_{\varphi} = \frac{4\alpha^2 k_{\xi}(1+u) + 1}{16\pi^2\alpha^4 r^4},$$

$$\delta T^{r}{}_{\varphi} = \alpha^2 r^2 \delta T^{\varphi}{}_{r} = -\frac{k_{\xi}\pi}{8\pi^2\alpha^2 r^3}.$$

They satisfy the local conservation and trace identities. At conformal coupling the correction is independent of (θ,χ) *and equals the one-mode conformal correction.*

Proof. In , take $V = v$, $H_W = 0$, and use equal orders. This gives $C = c_0 v v^{\dagger}$. The kernel is therefore with its angular mode replaced by w.

The necessary angular identities are

$$|w|^2 = 1 + u, \qquad |w'|^2 = \frac{1-u}{4}, \qquad w'' = -\frac{1}{4}w, \qquad w'\overline{w} - w\overline{w}' = \mathrm{i}\cos 2\theta.$$

Together with the radial second jet in the proof of Theorem , they give the displayed coincidence derivatives. At $\xi = 1/6$, substitution in gives $-[48\pi^2\alpha^4 r^4]^{-1}\mathrm{diag}\,(1,1,-3,1)$. For general ξ, apply twice the improvement operator to δF. In particular

$$\nabla_r \nabla_\varphi (r^{-2}(1+u)) = 3r^{-3}\pi,$$

which gives the off-diagonal entry. The Ward identities also follow from the smooth bisolution property, independently of these component formulas.

Scale-dependent diagonal nonnegative extensions

For $0 < \nu < 1$, $\lambda > 0$, define $M = M_\lambda$ by

$$(M/2)^{2\nu} = \lambda \frac{\Gamma(1+\nu)}{\Gamma(1-\nu)}.$$

Then

$$c_\lambda(p) = c_0(\nu)\frac{p^{2\nu}}{p^{2\nu} + M^{2\nu}}.$$

Theorem (Convergent observable integrals). *For one irregular deficient mode of charge q, put $\varepsilon = Mr$, $K = K_\nu(x)$. Then*

$$\delta F = \frac{\nu}{4\pi^2 \alpha r^2}\Phi_\nu(\varepsilon), \qquad \Phi_\nu(\varepsilon) = \frac{c_0}{\nu}\int_0^\infty \frac{x^{1+2\nu}K_\nu(x)^2}{x^{2\nu} + \varepsilon^{2\nu}}\,\mathrm{d}x,$$

$$\delta j_{\varphi} = -2eq\,\delta F, \qquad \delta j_{t} = \delta j_{r} = \delta j_{z} = 0,$$

and

$$\delta T^{a}{}_{b} = \frac{1}{4\pi^{2}\alpha r^{4}} \int_{0}^{\infty} x\,\mathrm{d}x\, \frac{c_{0} x^{2\nu}}{x^{2\nu} + \varepsilon^{2\nu}}\, t^{a}{}_{b}(x),$$

where

$$t^{t}{}_{t} = t^{z}{}_{z} = x^{2}K^{2} + (4\xi - 1)(\nu^{2}K^{2} + x^{2}K^{'}\wedge 2 + x^{2}K^{2}),$$

$$t^{r}{}_{r} = x^{2}K^{'}\wedge 2 - (x^{2} + \nu^{2})K^{2} + 4\xi x K K^{'},$$

$$t^{\varphi}{}_{\varphi} = 2\nu^{2}K^{2} - 4\xi x K K^{'} + (4\xi - 1)(\nu^{2}K^{2} + x^{2}K^{'}\wedge 2 + x^{2}K^{2}).$$

All off-diagonal entries vanish. Corrections from independent diagonal modes add.

Proof. Differentiate before coincidence. The Euclidean momentum angular average gives $[\partial_{\tau}\partial_{\tau'}] \mapsto p^{2}/2$. Radial differentiation and

$$K_{\nu}^{''} = -K_{\nu}^{'}/x + (1 + \nu^{2}/x^{2})K_{\nu}$$

reduce the point-split tensor to the displayed expressions. The exponential and small-momentum bounds of Theorem justify differentiation.

At fixed p, the relevant stress integrand is proportional to $r^{-2}t(pr)$. The Bessel equation gives

$$x(t^{r}{}_{r})^{'} - t^{r}{}_{r} = t^{\varphi}{}_{\varphi},$$

which is radial conservation before momentum integration. The trace follows from Lemma .

Theorem (Small-radius asymptotics with corrected remainders). *As* εdownarrow0, $\Phi_{\nu} \to 1$. *For* $0 < \nu < 1/2$,

$$\Phi_{\nu} = 1 - A_{\nu}\varepsilon^{2\nu} + R_{\nu}(\varepsilon),$$

where

$$A_{\nu} = \frac{c_{0}\sqrt{\pi}\,\Gamma(1-2\nu)\Gamma(1-\nu)}{\nu \qquad 4\Gamma(3/2-\nu)}$$

and

$$R_{\nu}(\varepsilon) = \begin{cases} O(\varepsilon^{4\nu}), & 0 < \nu < 1/3, \\ O(\varepsilon^{4/3}\log(1/\varepsilon)), & \nu = 1/3, \\ O(\varepsilon^{2-2\nu}), & 1/3 < \nu < 1/2. \end{cases}$$

At $\nu = 1/2$,

$$\Phi_{1/2}(\varepsilon) = 1 - 2\varepsilon e^{2\varepsilon}E_{1}(2\varepsilon) = 1 - 2\varepsilon\log(1/\varepsilon) + O(\varepsilon).$$

For $1/2 < \nu < 1$,

$$\Phi_{\nu} = 1 - B_{\nu}\varepsilon^{2-2\nu} + O(\varepsilon^{2\nu}),$$

where

$$B_{\nu} = \frac{c_{0}\,4^{\nu}\Gamma(\nu)^{2} \qquad \pi}{\nu \qquad 4 \qquad 2\nu\sin(\pi(1-\nu)/\nu)}.$$

The appropriately rescaled current and stress converge to their E_{0} *values.*

Proof. The bound by one in the coefficient and the integrability of xK_{ν}^{2} give dominated convergence. For $\nu < 1/2$, expand once:

$$\frac{1}{1+(\varepsilon/x)^{2\nu}} = 1 - (\varepsilon/x)^{2\nu} + \frac{(\varepsilon/x)^{4\nu}}{1+(\varepsilon/x)^{2\nu}}.$$

The first correction is the convergent Mellin integral defining A_{ν}. In the last term use $K_{\nu}(x)^{2} = O(x^{-2\nu})$ near zero and split at $x = \varepsilon, 1$. The remaining power integral is governed by $x^{1-6\nu}$, giving the three remainder regimes.

For $\nu > 1/2$, subtract the leading small-x expression

$$K_{\nu}(x)^{2} = a_{\nu}x^{-2\nu} + O(1), \qquad a_{\nu} = 2^{2\nu-2}\Gamma(\nu)^{2}.$$

The leading deficit is obtained by $x = \varepsilon y$ and

$$\int_0^\infty \frac{y^{1-2\nu}}{1+y^{2\nu}}\,\mathrm{d}y = \frac{\pi}{2\nu\sin\left(\pi(1-\nu)/\nu\right)}.$$

After this subtraction the remainder divided by $\varepsilon^{2\nu}$ is integrable both at zero and infinity, which proves the stated bound.

For $\nu = 1/2$, use $K_{1/2}(x)^2 = \pi e^{-2x}/(2x)$ and integrate directly. The stress limits follow by the same domination applied to the derivative integrands.

Theorem (Long-range tail). *As $r \to \infty$ at fixed $\lambda > 0$,*

$$\delta F = \frac{\nu}{4\pi^2\alpha(1+2\nu)\lambda\, r^{2+2\nu}} + O(r^{-2-4\nu}).$$

The leading stress coefficients are

$$\delta T^r{}_r = \frac{\nu[(1+2\nu) - 2\xi(1+\nu)(3+2\nu)]}{2\pi^2\alpha\lambda(1+2\nu)(3+2\nu)\, r^{4+2\nu}} + O(r^{-4-4\nu}),$$

$$\delta T^t{}_t = \delta T^z{}_z = \frac{\nu(1+\nu)[4\xi(1+\nu)(3+2\nu) - (2\nu^2+3\nu+2)]}{2\pi^2\alpha\lambda(1+2\nu)(3+2\nu)\, r^{4+2\nu}} + O(r^{-4-4\nu}),$$

$$\delta T^\varphi{}_\varphi = -(3+2\nu)\,[\delta T^r{}_r]_{\text{leading}} + O(r^{-4-4\nu}).$$

Where a leading coefficient is nonzero, its relative correction is $O((Mr)^{-2\nu})$. No relative assertion is made at a zero coefficient.

Proof. For $\varepsilon \to \infty$,

$$\frac{x^{2\nu}}{x^{2\nu}+\varepsilon^{2\nu}} = \varepsilon^{-2\nu}x^{2\nu} + O(\varepsilon^{-4\nu}x^{4\nu}).$$

All resulting K-Bessel moments, including the derivative moments, converge. Evaluate them with Appendix and the defining relation between M and λ. This gives δF and $T^r{}_r$. Conservation at power $r^{-4-2\nu}$ gives the angular coefficient; the trace identity gives the time coefficient. The same moment estimate supplies the absolute remainder.

The approach to Friedrichs is slow when ν is small: relative to r^{-2} or r^{-4}, the correction is of order $r^{-2\nu}$. This is the

power-law analogue of the logarithmic tail in . At fixed regular points the limits λdownarrow 0 and $\lambda \to \infty$ give E_0 and Friedrichs, respectively, by dominated convergence.

Rotation-breaking nonnegative relations

Every nonnegative boundary relation already has the convergent kernel . This section makes its observable content and asymptotic qualifications explicit.

Proposition (Matrix observables and radial current). *For $d = 2$, let $C = C_W$ and $K_j = K_{\nu_j}(pr)$. Then*

$$\delta F = \frac{1}{4\pi^2\alpha}\int_0^\infty p\,\mathrm{d}p \sum_{jk} C_{jk}K_jK_k e^{\mathrm{i}(q_j - q_k)\varphi},$$

$$\delta j_\varphi = -\frac{e}{4\pi^2\alpha}\int_0^\infty p\,\mathrm{d}p \sum_{jk} (q_j + q_k)C_{jk}K_jK_k e^{\mathrm{i}(q_j - q_k)\varphi}.$$

Writing the two labels as $0, -1$,

$$\delta j_r = -\frac{e}{2\pi^2\alpha}\int_0^\infty p^2\,\mathrm{d}p\,\mathrm{Im}\,(C_{0,-1}e^{\mathrm{i}\varphi})[K'_{\nu_0}(pr)K_{\nu_{-1}}(pr) - K_{\nu_0}(pr)K'_{\nu_{-1}}(pr)].$$

Also $j_t = j_z = 0$. Stress is obtained by applying to $2\mathrm{Re}\,\delta G_W$. All these operations converge locally and satisfy

$$\nabla_\mu T^\mu{}_\nu = 0, \qquad T^\mu{}_\mu = (6\xi - 1)\mathrm{Box}\,F,$$

$$\frac{1}{r}\partial_r(rj^r) + \partial_\varphi j^\varphi = 0.$$

Generically, unequal deficient orders and off-diagonal boundary data give a nonzero radial current and a nonzero shear component.

At $\beta = 1/2$, equal orders force the displayed radial current to vanish for every such relation. Other components may still have angular dependence.

Proof. Apply the derivative difference to . An angular derivative gives $\mathrm{i}(q_j + q_k)$. For a radial derivative combine the j, k and k, j terms using Hermiticity of C; this gives the imaginary part shown. At equal orders the radial Wronskian factor is zero. All interchanges are justified by Theorem . The Ward identities are Lemma .

Changing the finite-chart entry $H_{0,-1}$ to $e^{\mathrm{i}\gamma}H_{0,-1}$ changes the same phase in $C_{0,-1}$; it is equivalent to $\varphi \mapsto \varphi + \gamma$. Statements that every component must depend on angle, or that every non-diagonal relation must carry radial current, would ignore the equal-order exceptions.

Proposition (Finite positive matrices: large and small radius). *Suppose* $W = \mathbb{C}^2$ *and* $H > 0$. *Then*

$$\delta F \sim \frac{1}{4\pi^2\alpha}\sum_{jk}(H^{-1})_{jk}\frac{\nu_j\nu_k}{1+\nu_j+\nu_k}\frac{e^{\mathrm{i}(q_j-q_k)\varphi}}{r^{2+\nu_j+\nu_k}} \quad (r \to \infty),$$

with a termwise asymptotic expansion. The leading azimuthal-current terms are obtained by weighting the summands by $-e(q_j + q_k)$. *The off-diagonal power is* $r^{-2-1/\alpha}$.

For every full finite $H \geq 0$, *the small-radius leading rescaled observables are those of* $(\mathrm{E}_0, \mathrm{E}_0)$*:* $r^2\delta F$, $r^2\delta j_\varphi$, $r^3\delta j_r$, *and* $r^4\delta T$ *in an orthonormal frame have the corresponding limits.*

Proof. At small p, positive invertibility gives

$$C(p) = \frac{1}{2}E^{-1}H^{-1}E^{-1} + \text{higher powers of} p.$$

Set $x = pr$ and use

$$\int_0^\infty x^{1+\nu_j+\nu_k}K_{\nu_j}(x)K_{\nu_k}(x)\,\mathrm{d}x = \frac{2^{\nu_j+\nu_k-1}\Gamma(1+\nu_j+\nu_k)\Gamma(1+\nu_j)\Gamma(1+\nu_k)}{\Gamma(2+\nu_j+\nu_k)}.$$

Gamma reflection and recurrence give the displayed coefficient. A low/high momentum split, the Neumann expansion at small momentum, and exponential decay at high momentum justify the asymptotic integration and the derivative expansions.

At large p,

$$C(p) = (S + 2EHE)^{-1} \to S^{-1}$$

for every full finite H. After $p = x/r$, Lemma supplies an integrable dominating function for each rescaled observable, including its required derivatives. Dominated convergence gives the two-mode E_0 limits.

Remark (Boundary-chart and rank qualifications). The large-radius formula uses H^{-1} and is not asserted for singular H. The integral representation remains valid there.

The small-radius two-mode limit is a statement for a full *finite* chart. It is not valid for every boundary chart: the scale-invariant line family already provides a counterexample, because it is irregular in only one boundary direction. For arbitrary boundary relations use ; no unsupported universal endpoint interpolation is imposed.

The limits just stated are rescaled limits. They do not assert that divergent unscaled quantities converge to finite numbers.

Formula comparisons with the literature

The literature comparisons below concern stated formulas and translated conventions. They do not assert an exhaustive priority search or verification of an entire paper. The source access grades are defined in Section .

Friedrichs flux formulas

Sriramkumar's equations (2.12)–(2.16) [8] use $(+,-,-,-)$, radius ρ, and fractional flux γ. Set $\rho = r, \gamma = \beta$. Its lower-index current is our Theorem . For its minimally coupled complex tensor, the coefficients are

$$A = -C_\beta, \qquad B = -20P_\beta,$$

in

$$\frac{1}{720\pi^2 r^4}\{A\,\mathrm{diag}\,(1,1,-3,1) - B\,\mathrm{diag}\,(1,-1/2,3/2,1)\}.$$

With the covariant stress normalization held fixed, changing the metric sign reverses mixed components. This reproduces at $\xi = 0$. Its periodic-gauge two-point expression, equation (2.4), is twice our equal-time kernel after the gauge phase is included.

Spinelly–Bezerra de Mello [9], equations (30) and (36), agree with F after the flux-even identification $\gamma = \beta$. Their equation (37), as printed in the inspected version, is

$$\frac{1-\alpha^2-6\gamma(1-\gamma)}{48\pi^2\alpha^4 r^2}.$$

That display carries an extra α^{-2} relative to (30) and (36). It does not agree with *F*. The earlier sentence in this ledger, which copied (37) with denominator $48\pi^2\alpha^2 r^2$ and called it an agreement, was our transcription error. I-B-VP-4 records the same printed discrepancy as SB2. Their equations (54)–(56), in their displayed stress normalization, have half our complex multiplicity; their coupling term agrees after that translation.

Printed discrepancy retained. Their equation (55) prints $\gamma(1-\gamma^2)$, rather than $\gamma^2(1-\gamma)^2$, in the flux-dependent conformal term. The former is not symmetric under $\gamma \mapsto 1-\gamma$: it is 0.273 at 0.3 and 0.357 at 0.7. The latter is the polynomial obtained here and in Sriramkumar's equation (2.15). We identify an apparent typographical discrepancy in that displayed formula, not a demonstrated error in every result of the paper. No published erratum is asserted.

Bezerra de Mello–Bezerra–Saharian–Harutyunyan [10], equation (4.6), gives the zero-thickness massless current as an image sum plus an integral. For $D = 3$, $q = 1/\alpha$, their physical component $-j_2/r$, with $j_2 = j_\varphi/\alpha$, equals

$$\frac{eq^3\beta(1-\beta)(1-2\beta)}{12\pi^2 r^3}$$

in their signature convention. Their equation (4.7) is the $q = 1$ limit. The reported numerical comparisons are retained in Appendix ; they are not needed to prove the polynomial.

Guimarães [7] gives another explicit zero-temperature comparison. With $B = \alpha, \gamma = \beta, \rho = r$,

$$\omega_2 = \frac{P_\beta}{24\pi^2}, \qquad \omega_4 = \frac{C_\beta + 10P_\beta}{720\pi^2}.$$

Its equations (19) and (30) give $F = \omega_2/(2r^2)$ and, in order (t, z, r, φ),

$$\frac{\omega_4 - \omega_2/3}{r^4}\mathrm{diag}\,(1,1,1,-3) + \frac{4(\xi-1/6)\omega_2}{r^4}\mathrm{diag}\,(1,1,-1/2,3/2).$$

Reordering gives . This comparison uses the stated overlap of its integral presentation, $B > 1/2$; our all-angle result is proved independently.

Logarithmic boundary conditions and AKO

In Allen–Kay–Ottewill [13], their scattering length R and pole q map to our L and κ_L:

$$q = 2e^{-\gamma_E}/R.$$

Their equations (3.17), (4.3), and Appendix A give the $\nu = 0$ Krein coefficient and principal-value integral. Equation (4.8) agrees with the leading term of .

Their equations (4.13)–(4.15) are the one-real-field $\nu = 0$ versions of our diagonal stress integrands. The complex tensor here is twice those expressions. Their equation (4.17) is the same radial conservation identity.

AKO fit idealized exterior formulas to a smooth core. The literal zero-thickness logarithmic extension has an unstable band and no stationary finite-moment state by Theorem . The principal-value value is therefore not silently called its ground-state expectation.

Filgueiras–Moraes and other contexts

For $\beta = 1/2$, their von Neumann boundary parameter η_0 maps through the small-radius coefficients to

$$E_\perp = -\kappa_{\text{ref}} \left[\frac{\cos(\eta_0/2 + \pi/(8\alpha))}{\cos(\eta_0/2 - \pi/(8\alpha))} \right]^{2\alpha},$$

in the energy units used in their equation (10) [14]. That equation also includes the free longitudinal kinetic term. The ratio must select the real bound-state branch; in our boundary convention this is $\lambda < 0$. The reported eight-point comparison is retained below.

The inverse-square analysis of [12] supplies broader operator context. The fermionic references [18, 19, 20] concern different fields and boundary structures, not confirmations of the scalar tensor or current computed here.

The SAE source reports finding no direct scalar computation of all three observables for these zero-thickness $\mathrm{E}_0, \mathrm{E}_\lambda, U(2)$ choices. That is retained as a limited search report, not a theorem of novelty or an exhaustive literature claim.

Supported transfer does not transfer a state

For an angularly small chart

$$U \subset \{r > r_U,\ \varphi \in I\}, \qquad r_U > 0, \quad |I| < \pi,$$

with a fixed lift, define

$$T_{\mathrm{arr}}(x, x') = \min\big(\{\rho_0\} \cup \{R_k : k \neq 0,\ |\theta - 2\pi\alpha k| \le \pi\}\big).$$

The supplied Friedrichs transfer theorem gives

$$T_{\mathrm{arr}}(x, x') \ge L_U = 2r_U \sin(\pi\alpha/2) > 0$$

[42, lem:window, thm:transfer].

Assumption (Conjunctive supported-transfer contract). A comparison with a completion Green operation requires:

1. the angularly small chart, signature ledger, and parallel frame;

2. a declared compact regular reserve containing all retained shadows, initial-data regions, and intermediate source supports;

3. for each sign $\sigma = \pm 1$, the strict inequality

$$\sigma(t - t') < T_{\mathrm{arr}}(x, x')$$

on neighbourhoods of every compact pair support used in the operation and its estimates.

For example a flat retained-past shadow may obey

$$J^{-}_{\eta}(\overline{\mathcal{O}}) \cap ([0, T_{\mathrm{lab}}] \times \mathbb{R}^3) \mathrm{Subset}\, \mathrm{dev}(U).$$

A sufficient ball-laboratory condition is

$$T_{\mathrm{lab}} < \min\{R_{\mathrm{lab}} - r_{\mathrm{lab}}, L_U\},$$

together with time and regularity reserves. The two bounds protect different requirements [44, 45].

Equality of retarded and advanced operations in such a supported window fixes a commutator there, not a symmetric covariance. It does not imply $W_F = W_{\text{flat}}$. The nonzero smooth Wick-square remainder is an explicit counterexample to that inference.

The direct Hadamard singularity is identified by Theorems –, not by transferring a completion state. No supported-transfer theorem for a different extension is assumed without a corresponding proof.

Finite reporting, evidence grades, and integrity

Proposition (Complete destructive report instrument). *Let $C \cong \mathbb{C}^D$, $D \geq 2$, have calibrated orthonormal basis $\{|\tau_j\rangle\}$, and let A contain all surviving finite registers. Install finite words w_j, retaining j even when words coincide. Put*

$$D_j = \langle \tau_j|_C \otimes I_A, \qquad M(X) = \sum_j \text{lvertj}, w_j\rangle\text{langlej}, w_j|_R \otimes D_j X D_j^\dagger.$$

Then M is completely positive and trace preserving,

$$\text{Tr}_R M(X) = \text{Tr}_C X,$$

and the identities persist with an untouched finite reference. There is no output quantum C-register.

Proof. The Kraus maps are $L_j = \text{lvertj}, w_j\rangle \otimes D_j$, and

$$\sum_j L_j^\dagger L_j = \sum_j |\tau_j\rangle\langle\tau_j| \otimes I_A = I.$$

Writing $X = \sum_{jk} |\tau_j\rangle\langle\tau_k| \otimes X_{jk}$, the output blocks are X_{jj}, whose sum is $\text{Tr}_C X$. The proof is unchanged with a reference factor.

This is the finite interface discussed in [41, 49, 47]. It reports installed words; it does not evaluate an unknown continuum covariance. The states $(|\tau_0\rangle \pm |\tau_1\rangle)/\sqrt{2}$ have identical outputs, so there is no universal left inverse. The record marginal also need not determine the surviving state.

Proposition (Finite-test limitation). *A finite family of compact linear evaluations need not determine a conserved smooth symmetric tensor on a regular flat region. If an independently specified finite-dimensional residual class $\mathcal{V}$ contains every residual admissible for a claim, an evaluation map injective on $\mathcal{V}$ does determine residuals within that class.*

Proof. For $h \in C_c^\infty$ in a flat chart set

$$S_{\mu\nu}[h] = (\partial_\mu\partial_\nu - g_{\mu\nu}\mathrm{Box})h.$$

It is symmetric and conserved. Finitely many evaluations impose finitely many linear conditions on the infinite-dimensional space of h's. Choose a nonzero h in their common kernel. The map $h \mapsto S[h]$ is injective on compact supports: if $S[h] = 0$, tracing gives $\mathrm{Box}\, h = 0$, then its Hessian is zero, so h is affine and compact support forces $h = 0$. Thus the evaluations miss a nonzero conserved tensor. The finite-dimensional statement is injectivity itself.

This is a coverage distinction, not a construction of a difference of two admissible Green kernels [48, 42].

Evidence and access grades

\texttt{ANALYTIC} A result proved by the displayed argument, with cited analytical inputs identified.

\texttt{SUPPLIED_\allowbreak NUMERIC} An execution statement inherited from one of the supplied records. It has not been rerun in preparing this consolidated text.

\texttt{UNCERTIFIED} A computation lacking the artifacts and rigorous error enclosure needed for a certified numerical interpretation.

All numerical reports in the appendices are simultaneously by provenance and as total-error bounds. An inherited symbolic execution report is not a newly executed symbolic proof.

The bibliography's marks preserve the predecessor access ledger: means the source reported reading the full paper; means metadata, sometimes an abstract, or citations in a read paper were consulted without reading that full paper; means recalled from memory. These are archival access grades, not claims that the present consolidation re-read

every marked work in full. The formula-level comparisons identify the passages actually relevant to the mathematics.

A verification gate. A reproducible numerical package should record

$$\begin{gathered}(\text{formula/version},\ \text{parameter encodings},\ ?\text{frame and lift},\ \text{evaluation pair},\\ \text{raw compared values},\ \text{working precision},\ ?\text{quadrature and truncation settings},\\ \text{tail bounds},\ \text{absolute residual},\ ?\text{relative residual},\ \text{status}).\end{gathered}$$

A relative residual is inapplicable when its denominator is zero or unreliable. A successful retention of a known discrepancy does not remove that discrepancy.

Agreement at a shared rounded binary input is not equal accuracy at an intended irrational parameter. For example,

$$\partial_\alpha F_\beta = -\frac{1-6\beta(1-\beta)}{24\pi^2 r^2 \alpha^3}$$

gives a separate input-error estimate from a parameter enclosure.

For a generic-angle diffraction tail with $|\Phi_\beta| \le Ce^{-\delta u}$, $u \ge U$,

$$|\text{tail}| \le \frac{Ce^{-(1+\delta)U}}{4\pi^3 \alpha r_{\min}^2 (1+\delta)}.$$

A certified run must supply C, δ, U, and differentiated bounds for stress or current; merely invoking exponential decay is not a certificate.

Execution and build status. The source files name scripts and logs, but their contents were not supplied with the manuscripts. No independent numerical run, interval verification, proof-assistant check, or successful pdflatex run is asserted here. This manuscript is self-contained as a LaTeX source, not as a reproduction archive for those computations.

Unified extension and state ledger

Throughout this table $m = 0$, unless a row says otherwise. ``Constructed'' refers to the declared spectral prescription, not universal uniqueness of algebraic states. ``Evaluated by integrals'' means a proved convergent

formula, not a claim of elementary closed form.

begingroup 1.16 toprule Extension or construction Status Formula, proof, or limitation

midrule endfirsthead toprule Extension or construction Status Formula, proof, or limitation

midrule endhead bottomrule endfoot

Friedrichs, $\beta = 0$ Constructed; closed forms Infrared Lemma ; Theorem . Only nonnegative extension in the untwisted massless product ledger, not uniqueness of every algebraic state.

Friedrichs, $0 < \beta < 1$ Constructed; closed forms Theorems and ; all cone angles, including image-boundary parameters.

No deficient modes Friedrichs only The transverse minimal operator is essentially self-adjoint.

L_{L}, every finite $L > 0$ Excluded as stationary finite-moment state One negative transverse eigenvalue and an unstable full band; Theorems and .

Entire transverse bound channel removed Projected construction; not local Hadamard Extra logarithmic singularity and divergent subtracted square; Proposition (a). Projected CCR.

Only negative full spectrum removed Projected Wick square evaluated; stress prescription not supplied Principal-value Wick square; nonsmooth direction-dependent remainder and boost breaking; Proposition (b). Not the original local field.

E_0 in one deficient mode Constructed; closed forms Theorem , with the legitimate shifted branch. Replaces predecessor ``Open'' status.

$(\mathrm{E}_0, \mathrm{E}_0)$ Constructed; closed forms Additive formula in Theorem .

Diagonal $\lambda_j \in [0, \infty]$ Constructed; convergent integrals Theorems , –. Finite-λ endpoint and asymptotic qualifications apply.

Any diagonal combination with a negative λ_j Excluded as stationary finite-moment state Theorems and .

Half-holonomy $\mathbb{CP}^1$ line family Constructed; closed forms Theorem . Rotation breaking is allowed; radial current vanishes.

Full finite $U(2)$ chart, $H \geq 0$ Constructed; integral evaluation Theorems , ; Propositions , . Large-radius formula needs $H > 0$.

Nonnegative boundary charts Constructed; integral evaluation Compressed coefficient . The full finite-chart two-mode ultraviolet limit is not imposed on these relations.

Any relation with $H_W \not\geq 0$ Excluded as stationary finite-moment state Theorems and .

Massive extensions, including L_L with $m \geq \kappa_L$ **Open** here A massive baseline and covariance analysis are not supplied. At equality, nonnegativity is insufficient because of the longitudinal threshold infrared divergence.

Non-product or longitudinal-momentum-dependent extensions **Open** Boundary data are infinite dimensional and require an operator-valued relation and Weyl function. Generic choices need not preserve longitudinal boosts.

Geometrically twisted tensor sectors, spinors, gauge fields **Open** No spin structure, constrained gauge domain, or multiplet state is constructed.

Thermal and other non-ground states **Open** Not constructed. Their infrared conditions are separate from the ground-state estimates.

Apex-supported extensions of the observables **Open** No distributional completion or apex counterterm class is chosen.

Global microlocal spectrum analysis **Open** Local Hadamard form is proved; global diffractive propagation is not classified here.

endgroup

Remark (Massive threshold warning). If a transverse bound eigenvalue is $-\kappa^2$ and $m = \kappa$, its longitudinal dispersion is $|p_z|$. A test with nonzero zero-momentum projection then gives

$$\int_{|p_z|<1} \frac{|\hat{v}(p_z)|^2}{2|p_z|}\,\mathrm{d}p_z = \infty.$$

Thus nonnegativity at a massive threshold does not guarantee the ordinary covariance on every compact test.

Remaining mathematical questions. The unclosed items are:

1. completion-supported Wick-square, current, and stress extensions and their counterterms;

2. global microlocal propagation for the selected states;

3. massive states and observables, including threshold restrictions;

4. induced Maxwell fields and broader source-by-source literature audits;

5. a smoothed-core or distributional apex curvature-coupling model selecting an effective extension;

6. general non-product extensions;

7. spinors and constrained gauge systems;

8. thermal and other non-ground states;

9. simpler closed stress formulas for generic matrix relations.

The first and fifth questions are not answered by the vanishing regular-part curvature anomaly. On the declared regular manifold, ξ changes the stress expression but not the fixed scalar equation or chosen extension.

Non-identifications and conclusion

The cone, deficit, extension, spectral state prescription, ξ, and β are declared inputs. None is derived from an Enter record. No physical preference among the nonnegative extensions is claimed, and no zero-thickness relation is asserted to model a particular real string core.

A symmetric, conserved, traceless tensor need not source the prescribed background. For example, in a regular flat chart

$$t_{\mu\nu} = \mathrm{diag}\,(3e_0, e_0, e_0, e_0), \qquad e_0 > 0,$$

is constant, conserved, and traceless, while the local Einstein tensor is zero. Equivalently a constant null covector gives the witness $k_\mu k_\nu$. This is the I-B prop:not-source separation [41, 43].

We do not insert the expectation values into Einstein or Maxwell equations, compute a sourced magnetic field, solve backreaction, or claim spacetime emergence. Negative energy is a property of the stated field, state, and normalization, not a certification of a programme direction.

The observables are smooth tensor or scalar fields, hence distributions, on M. Their generic r^{-2}, r^{-3}, and r^{-4} behaviours are not ordinary locally integrable completion data with transverse measure $r\,\mathrm{d}r$. Distributional extensions may exist, but require a prescription and may have apex-supported ambiguities. For finite nonnegative charts the leading diagonal ultraviolet powers are the scale-invariant ones; singular one-endpoint boundary data must not be confused with a universal stronger simultaneous diagonal power.

Local point splitting does not establish an interacting algebra, time-ordered product, quantum Ward identity, BRST closure, graviton or ghost sector, or apex gauge condition. A finite reporting instrument is not a continuum covariance, and finite numerical tests are not a coverage theorem.

No companion status record is updated. In particular:

• no new closure of O.1 is asserted outside any independently declared scoped result;

• any status outside such a scope remains unchanged;

• the frozen XVI-N insertion is not discharged;

• XV-B's downstream quantum-admission exit is not opened;

• is not promoted;

• no XVI-R or frozen-tree release is asserted. These are non-transfer boundaries, not premises of the scalar proofs [47, 50, 51, 52].

The supplied XIV-H/I/J and XV-B excerpts distinguish constructed internal kernels or off-characteristic references from an unconstructed complete characteristic quantum map. Nothing here identifies our scalar extension problem

with that graph problem or supplies its missing admission. A registered technical manuscript named XV-B is not declared nonexistent; ``not opened'' refers only to the stated downstream admission gate.

Conclusion. Within the declared massless spatial-product class, every extension has a precise positivity and state status. Nonnegative relations have infrared-finite spectral covariances, local Hadamard form, and convergent regular-part observable formulas. Scale-invariant cases admit explicit closed forms. Unstable relations are excluded by a local compact-test stationarity contradiction, while their projected substitutes are identified as different, nonlocal constructions. The familiar Friedrichs results, the complex first-jet current, and all retained computational reports are presented with separate analytical and evidential status. No completion, dynamics, or programme closure follows from them.

appendix

Integral identities and asymptotic calculations

For real μ, ρ and $s > |\mu| + |\rho|$,

$$\int_0^\infty x^{s-1} K_\mu(x) K_\rho(x)\,\mathrm{d}x = \frac{2^{s-3}}{\Gamma(s)} \Gamma\Big(\frac{s+\mu+\rho}{2}\Big)\Gamma\Big(\frac{s+\mu-\rho}{2}\Big)\Gamma\Big(\frac{s-\mu+\rho}{2}\Big)\Gamma\Big(\frac{s-\mu-\rho}{2}\Big).$$

One derivation inserts the Schwinger representations for the two K-functions, integrates the Gaussian power of x, then sets the two positive Schwinger parameters equal to $tv, t(1-v)$. The t-integral is a gamma integral and the v -integral is a beta integral. The stated condition justifies absolute convergence. This also provides a direct proof of the identity used in the asymptotic sections; see [22, 23, 24] for the standard special-function framework.

In particular,

$$\int_0^\infty x K_v(x)^2\,\mathrm{d}x = \frac{\pi v}{2\sin \pi v}, \qquad 0 < v < 1,$$

and at $v = 0$ the continuous limit is 1/2. Differentiation in s at $s = 2, \mu = \rho = 0$ gives

$$\int_0^\infty x K_0(x)^2 \log x\,\mathrm{d}x = \frac{1}{2}(\log 2 - 1 - \gamma_E).$$

For derivative moments, integration by parts gives, whenever the endpoint terms vanish,

$$\int_0^\infty x^{a+1} K_\nu K_\nu' \, \mathrm{d}x = -\frac{a+1}{2} \int_0^\infty x^a K_\nu^2 \, \mathrm{d}x.$$

The Bessel equation then reduces K'^2 moments to the same Mellin family. These operations yield the coefficients of Theorem without an appeal to a symbolic run.

Boundary-data normalization. The identities

$$d_\nu e_\nu = \frac{1}{2\nu}, \qquad \Delta_\nu = \nu s_\nu d_\nu / e_\nu, \qquad 2e_\nu^2 \Delta_\nu = s_\nu$$

are the normalization checks behind . They prevent a second angular normalization factor from being inserted into . That equation uses unnormalized exponentials in its displayed sum; uses the normalized ψ_j's.

Retained Friedrichs computational reports

Every execution statement in this and the following appendices is and . The original scripts and raw logs were not included in the supplied text. No run is asserted here.

The named files are

$$\text{nolinkurl}\, code/symbolic.py, \quad \text{nolinkurl}\, code/numerics.py, \quad \text{nolinkurl}\, code/numerics_\nu 2.py,$$

$$\text{nolinkurl}\, logs/symbolic.json, \quad \text{nolinkurl}\, logs/numerics.json, \quad \text{nolinkurl}\, logs/numerics.log, \quad \text{nolinkurl}\, logs/numerics_\nu 2.json.$$

The sources report SymPy 1.14 and mpmath 1.3, initially 30-decimal-digit work and subsequently 40 digits, with parameters and coordinates entered as IEEE doubles. These are inherited provenance descriptions, not available execution artifacts.

To distinguish duplicated source labels, this appendix calls the Friedrichs symbolic suite F-S1–F-S4. The SAE suite retains SAE-S1–SAE-S4 below.

Symbolic reports F-S1–F-S4

F-S1 The geometric mode sum and time-split expansions reportedly matched the displayed Wick-square and stress formulas. The log keys phi2_beta0_matches_lit, Ttt_beta0_xi16_matches_lit, and Ttt_beta0_general_xi_matches_FS were reported true. The corresponding identities have the independent proof in Lemma .

F-S2 For $N = 2, \dots, 6$, $\alpha = 1/N$, and $\xi = 0, 1/6, 1/4$, the full Cartesian image tensor reportedly matched exactly in 15 cases: 60 diagonal and 180 off-diagonal entries. All off-diagonal entries and the coincidence on-shell residual were reported zero.

As $1440\pi^2 r^4 (T^t_{\ t}, T^r_{\ r}, T^\varphi_{\ \varphi}, T^z_{\ z})$, the retained examples are

$c\|c\|cN$	ξ	reported vector
hline 2	1/6	$(15, 15, -45, 15)$
3	1/6	$(80, 80, -240, 80)$
5	0	$(144, 864, -2592, 144)$
5	1/4	$(864, 504, -1512, 864)$

They also follow directly from .

F-S3 The $\csc^2$ and $\csc^4$ identities were reportedly checked exactly for $N = 2, \dots, 8$. Proposition proves the stronger all-N identities.

F-S4 Conservation, trace, and $(g\mathrm{Box} - \nabla\nabla) r^{-2}$ were reportedly checked symbolically. They are proved analytically in Lemma and Theorem .

N1: off-diagonal contour comparisons

For each of

$$\alpha \in \left\{ 1 - \frac{1}{2\pi}, ?1 - \frac{\sqrt{2}}{2\pi}, ?\frac{3}{4}, \frac{1}{\sqrt{2}} \right\}, \qquad \beta = 0,$$

and for

$$(\alpha, \beta) = \left(1 - \frac{1}{2\pi}, 0.3 \right), \quad (0.4, 0.3),$$

the source reports three pairs and a maximum absolute difference 9.5×10^{-19} between contour and mode values of order 10^{-2} to 1.

The source attributed this scale to double-precision chart inputs. Without the raw paired values and error propagation this is a reported diagnostic, not a verified cause.

N2: coincidence Wick square and revised diagnostics

The source reports relative differences at most 1.1×10^{-31} for a set containing

$$\alpha = 1 - \frac{1}{2\pi}, ?1 - \frac{\sqrt{2}}{2\pi}, ?\frac{1}{\sqrt{3}}, \frac{3}{4}$$

and two twisted cases with $\beta = 0.3$.

For a second set containing

$$\alpha = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, 0.3, \qquad (\alpha, \beta) = (0.4, 0.7),$$

the retained earlier residual is 8.6×10^{-17}.

The AB record reports that image terms had been cast to double complex. It further reports an omitted diffraction correction at a double approximation to 1/3. After removing the cast, it reports N2 discrepancies at most 4×10^{-41}, except $\alpha =$ texttt *double*(1/3), for which the omitted term reportedly leaves 5.8×10^{-17}. It reports N2/N3 reproduction at at most 6×10^{-39}. All of these numbers and explanations are retained; none replaces a raw log or an error enclosure.

Two supplied values of $r^2 F_0$ are

$$8.7471068907447577 \times 10^{-4} \quad (\mu = 1),$$

$$1.6886863940389629 \times 10^{-2} \quad (\alpha = 1/3).$$

The exact second expression is $8/(48\pi^2)$. Pure diffraction at $\alpha > 1/2$ and compatible finite images at exact quotient

parameters are analytical conclusions, not consequences of those decimals.

N3: full tensor residuals

The source describes symbolic second differentiation followed by quadrature, for

$$\alpha \in \left\{ 1 - \frac{1}{2\pi}, ?1 - \frac{\sqrt{2}}{2\pi}, ?\frac{1}{\sqrt{3}} \right\}, \quad \beta = 0,$$

and $(\alpha, \beta) = (1 - 1/(2\pi), 0.3)$, with $\xi = 0, 1/6, 1/4$ and $r = 1, 2$, giving 24 runs.

3pt toprule α, β ξ $1440\pi^2 r^4 (T^t{}_t, T^r{}_r, T^\varphi{}_\varphi, T^z{}_z)$ rel. diff. tr T r^4

midrule $1 - 1/(2\pi), 0$ 1/6 $(1.000488680294, \cdot, -3.001466040882, \cdot)$ $4.6\,10^{-31}$ $-2.5\,10^{-34}$

$1 - 1/(2\pi), 0$ 0 $(-7.28723785, 5.14435194, -15.4330558, \cdot)$ $1.8\,10^{-31}$ $-1.74942\,10^{-3}$

$1 - \sqrt{2}/(2\pi), 0$ 1/6 $(1.773131450488, \cdot, -5.319394351463, \cdot)$ $1.9\,10^{-31}$ $-1.9\,10^{-34}$

$1/\sqrt{3}, 0$ 1/6 $(8, 8, -24, 8)$ $2.3\,10^{-31}$ $-3.9\,10^{-34}$

$1/\sqrt{3}, 0$ 1/4 $(28, -2, 6, 28)$ $9.8\,10^{-32}$ $4.22172\,10^{-3}$

$1 - 1/(2\pi), 0.3$ 1/6 $(-3.292315687470, \cdot, 9.876947062410, \cdot)$ $6.2\,10^{-31}$ $1.9\,10^{-34}$

bottomrule A dot means equality to the time component.

The reported maxima are:

$l\|l$relative tensor difference	6.2×10^{-31}
off-diagonal magnitude	1.4×10^{-41}
dimensionless radial conservation residual	1.5×10^{-33}
on-shell coincidence residual	7.7×10^{-34}

The conformal trace was reported of order 10^{-34}, and the $r = 1, 2$ values reportedly obeyed r^{-4} scaling.

The original prose stated the untwisted real trace for every nonconformal run. That is not the correct comparator for a twisted complex run. The latter must use

$$\left(\xi - \frac{1}{6}\right) \frac{P_\beta}{2\pi^2 r^4}.$$

The retained untwisted example $-1.74942137814895 \times 10^{-3}$ at $\alpha = 1 - 1/(2\pi)$, $\xi = 0$, is compatible with the real trace formula.

N4: small-deficit slopes

At $r = 1, \beta = 0$, with stress at $\xi = 1/6$: toprule μ F_0/μ T^t_t/μ

midrule 10^{-1} $6.88294860341 \times 10^{-4}$ $4.66344397439 \times 10^{-5}$

10^{-2} $6.73514437791 \times 10^{-4}$ $4.49725955867 \times 10^{-5}$

10^{-3} $6.72067407377 \times 10^{-4}$ $4.48116263845 \times 10^{-5}$

midrule exact limit $1/(48\pi^3)$ $1/(720\pi^3)$

reported decimal $6.71906967358 \times 10^{-4}$ $4.47937978239 \times 10^{-5}$

bottomrule This is compatible with the proved pointwise slopes; it is not a uniform-limit proof.

Retained AB checks A1–A5

The named additional files are

$$\text{nolinkurl}\, code/ab.py, \quad \text{nolinkurl}\, code/ab_common.py, \quad \text{nolinkurl}\, logs/ab.json.$$

The source reports 40-digit work at shared IEEE-double inputs.

A1: exact phased images

For $N = 2, \ldots, 6$, $\alpha = 1/N$, and $\beta = j/N$, $j = 1, \ldots, N-1$, the source reports 15 exact symbolic cases. The reported quantities are the Wick square, all current components, and conformal time stress. The retained scaled examples are

$$\left(48\pi^2 r^2 F, ?12\pi^2 r^2 j_\varphi/e, ?720\pi^2 r^4 T^t_t\right) :$$

$c\|c\|cN$	β	reported triple
hline 3	1/3	$(-4, -2/3, -40)$
4	1/4	$(-3, -3/2, -15)$
5	1/5	$(0, -12/5, 144)$
6	1/6	$(5, -10/3, 545)$
6	1/2	$(-19, 0, -1135)$

The conjugate pairs reportedly have equal Wick squares and stress and opposite current. The quotient identity and the closed formulas provide independent analytical checks.

A2: irrational-angle contour checks

The reported parameter pairs are

• $(1 - 1/(2\pi), 0.3)$, $(1 - 1/(2\pi), 0.8)$;

• $(1 - \sqrt{2}/(2\pi), 5^{-1/2})$;

• $(3^{-1/2}, 0.3)$, $(3^{-1/2}, 0.45)$, with reported $\mu = 2.656$;

• $(0.3, 0.37)$, $(\pi^{-1}, 0.7)$, where nontrivial images occur. The reported maximum relative differences are 3.1×10^{-41} for F and 2.1×10^{-40} for j_{φ}. The other current components are reported zero.

Two retained current values are

$$r^2 j_{\varphi}/e = -1.00315107702538829175976 2 \times 10^{-3}$$

at $(1 - 1/(2\pi), 0.3)$, and

$$r^2 j_{\varphi}/e = +7.000000000000000 \times 10^{-3}$$

at $(\pi^{-1}, 0.7)$. At the exact latter parameters the polynomial value is 0.007. The full twisted tensor is the N3 case above; the AB source reports its revised reproduction at 6×10^{-39}.

A3: symmetries

The source reports

$$|F(\beta) - F(1-\beta)|, \quad |j(\beta) + j(1-\beta)| \le 7 \times 10^{-19},$$

and exact zero at $\beta = 1/2$. It attributes the residual to forming $1 - \beta$ in double precision. It also reports off-diagonal mode periodicity to 3.6×10^{-17}, attributed to the double rounding of 1.3. These attributions remain unverified diagnostics.

A4–A5: external formula comparisons

The source reports current comparisons with [10] at eight pairs whose approximate $q = 1/\alpha$ values are

$$1.189,\ 1.290,\ 1.732,\ 1.732,\ 3.333,\ \pi, ?3.846,\ 1.923.$$

It reports relative differences at most 1.2×10^{-39}, except 2.9×10^{-31} at $q = 1.923$, attributed to quadrature near a small denominator. The $q = 1$ limit at $\beta = 0.1, 0.3, 0.45, 0.7$ reportedly agrees to 5×10^{-38}.

Against [8], it reports 2×10^{-43} for the signature-translated minimal tensor and 1.2×10^{-44} for the current. It also reports exact agreement with the coupling term of [9] and agreement with its conformal term only after replacing the printed flux factor as discussed in Section . That reported agreement does not cover the printed equation (37), which carries the extra α^{-2} corrected above.

The source states that full texts were stored in lit/. That directory was not supplied here.

Retained SAE checks

The named files are

nolinkurl *code/sae*$_c$*ommon.py*, ?nolinkurl *code/sae*$_s$*ym.py*, ?nolinkurl *code/sae*$_n$*um.py*, ?nolinkurl *code/sae*$_l$*ead.py*, ?nolinkurl *code/sae*$_f$*m.py*,

with logs/sae_*.json, including logs/sae_lead.json and logs/sae_fm.json. The source reports 30-digit work unless stated otherwise and a double-exponential quadrature rule tested on $\int xK_\nu^2$ to 4×10^{-29}.

The source numbering skips C6. No missing C6 result is invented.

SAE-S1 The exact E_0 kernel reportedly gives the shifted Wick square, current, and four diagonal stress formulas

with residual exactly zero in both modes. Conservation, trace, current conservation, and diagonality were reported exact. Theorem supplies the independent proof.

SAE-S2 For the half-holonomy line family, conservation, trace, and current conservation were reported exactly zero. The compact forms reportedly agreed at rational sample points to 8×10^{-174}, and $\theta = 0$ reproduced the single-mode formulas. Sampled high precision is not an exact identity certificate; Theorem gives the analytical calculation.

SAE-S3 The Krein stress integrands were reported to satisfy pointwise conservation numerically below 10^{-166} (the abstract also reported 10^{-167}). Their $v = 0$ translation to AKO (4.13)–(4.15) was reported with symbolic residual zero. The Bessel identity in Theorem proves the conservation relation.

SAE-S4 A truncated comparison of the branch series $S(\beta \pm 1)$ was reported with residual 10^{-63}. It is a truncation report, separate from the exact branch identity in Theorem .

C1 Ten Krein-integral versus E_0 closed-form cases reportedly included irrational α, β, mode 0, mode −1, and both, with $\xi = 0, 1/6, 0.25, -0.1$. The maximum relative deviation was 5.9×10^{-30}. Reported off-diagonal stress was at most 2×10^{-36}, radial current at most 10^{-65}, and differentiated conservation, trace, and current residuals at most 7×10^{-31}.

C2 An eigenfunction expansion

$$\int_0^\infty k\,\mathrm{d}k \frac{f_k(r) f_k(r') - J_\nu(kr) J_\nu(kr')}{k^2 + p^2}, \qquad f_k \propto t(k) J_\nu + J_{-\nu},$$

was reportedly compared with the Krein term at $r \neq r'$. Four E_0 or $\lambda > 0$ cases gave 1.1×10^{-15} at 20 digits, described as limited by an oscillatory tail.

C3 For matrix relations the reports give: Hermiticity residual 3×10^{-47}; boundary-condition residual 10^{-42} from the $I_{\pm\nu}$ split and 6×10^{-16} from a direct small-radius fit; and positive C in the tested nonnegative cases. For 12 random Hermitian matrices, the reported determinant-root count agreed with the number of negative matrix

eigenvalues, including zero, one, and two transverse bound states. Theorem proves the count without random tests.

C4 At $\alpha = 0.8, \beta = 0.3$, the tested orders were $\nu = 0.375, 0.875$. The source reports small-ε asymptotic relative errors 4.5×10^{-3} and 2.6×10^{-9} at $\varepsilon = 10^{-6}$. For large $\varepsilon = 10^{6}$, it reports 2.8×10^{-5} and 5.8×10^{-11}. It reports $j/F = -2eq$, agreement of leading stress coefficients to 12 printed digits, structural relations to 10^{-29}, and conservation and trace residuals at most 10^{-30}. The conjugate-mode swap reportedly agrees in square and stress to 3.5×10^{-30}, with opposite current.

The small-radius tests lie in regimes compatible with the corrected Theorem ; they do not establish the incorrect predecessor remainder uniformly for all $0 < \nu < 1/2$.

C5 For $\alpha = 0.83, \beta = 0.45$, and a positive matrix with reported eigenvalues $0.49, 2.11$, the source reports conservation 1.5×10^{-31}, trace 7×10^{-31}, current conservation 1.3×10^{-32}, and $j_r = -1.63 \times 10^{-5} \neq 0$. The rotation rule was reported exact. The off-diagonal-to-zero limit reportedly agrees with the diagonal result to 4×10^{-31}, and the line family integral with its closed form to 6×10^{-30}.

The SAE text additionally reports large-radius relative error 5×10^{-4} at $r = 10^{3}$, described as decreasing like $r^{-2\nu_0}$, and a small-radius scaled error 2×10^{-4} at $r = 10^{-6}$. These tests are not extended to arbitrary boundary charts.

C7 The reported bound-state diagnostics are:

• after pole subtraction, values -0.04893705 on both sides of $p = \kappa(1 \pm 10^{-8})$, differing by 1.6×10^{-9};

• an algebraic p^{-2} tail coefficient $-\kappa^2 K_0 K_0/(\pi\alpha)$, agreeing to 3×10^{-5} at $p = 200$;

• the Struve identity to 3×10^{-26}, the longitudinal split identity to 5×10^{-15}, and the principal-value Hankel identity below 10^{-25};

• the E_λ bound-state residue matching lVertK_$\nu\|^{-2}$ to 12 digits;

• logarithmic asymptotic tests at $\kappa_L r = 10^{\pm 3}, 10^{\pm 6}, 10^{\pm 12}$, with two-term relative error 2.2×10^{-3} at 10^{12}, described as

decreasing like $\log^{-2}$.

Filgueiras–Moraes report Eight (α, η_0) comparisons were reported at most 6×10^{-19}. The energy scale and longitudinal term must be restored when comparing the dimensionless expression with their equation (10).

What would strengthen these reports. Supply the actual scripts, exact parameter encodings, dependency versions, raw values, precision schedules, quadrature tolerances, oscillatory truncations, pole prescriptions, certified tails, and image-boundary handling. Until then the retained numbers are comparisons, not rigorous error enclosures or proof of a universal identity.

Technical and documentary clarifications

1. Product completeness. The finite ledger is complete for Assumption , not for every self-adjoint extension of the full three-dimensional minimal operator. Non-product boundary data are infinite dimensional.

2. Positivity and infrared behaviour. The nonnegative extension condition, the compact-test covariance estimate, and thermal infrared behaviour are separate questions.

3. Boundary relations. A finite Hermitian matrix is one chart, not the entire extension space. Compression in supplies the remaining charts.

4. Transverse versus longitudinal spectrum. A transverse bound eigenfunction gives a longitudinal continuous branch. The instability proof uses that branch rather than an invented full-space eigenvector.

5. Analytic branch versus physical flux periodicity. The shifted formulas refer to a fixed entire geometric branch. They do not eliminate the extension change by integer flux relabeling.

6. Complex first jet. Taking a real part is appropriate for the stress prescription but destroys the current information if done before its first jet.

7. All diagonal cone parameters. Joint ellipticity proves the regular representative even where individual image sectors jump.

8. Projection versus original CCR. The principal-value reference is not positive; the projected positive covariances have projected commutators and do not solve the original local-state problem.

9. Completion terms. Vanishing local curvature ambiguities on M does not classify worldsheet-supported terms or stress-correlation contacts.

10. Transfer versus covariance. Supported equality of Green operations does not identify their symmetric states.

11. Finite arrays versus continuum normalization. A finite report, finite normal-jet array, or small numerical residual is not automatically a finite-dimensional exhaustive normalization class or a continuum identity.

12. External programme boundaries. The supplied XIV-H/I/J and XV-B excerpts describe different objects, fields, domains, contacts, and quantum gates. Internal retarded-kernel reconstruction, characteristic composition, output wavefront control, and quantum admission remain distinct there. No theorem here supplies a missing graph product or promotes a classical master-equation result to quantum closure.

13. Verification status. No successful compilation, independent execution, machine-checked proof, referee acceptance, or journal acceptance is claimed. The mathematical assertions released here are the displayed proofs with their explicit hypotheses.

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