I-B-VP-1: Prescribed apex finite parts on a conical Enter spacetime

Wick square, stress tensor, and charged current: strong extension, Ward identities, and classified contact freedom

Version 3: general $U(2)$ extensions, physical cores, worldsheet fields, the Dirac field, the $\lambda\varphi^4$ tadpole, and a re-audit of version 2

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Abstract

We construct distributional extensions across the apex worldsheet $\mathscr{W}$ of a conical Enter spacetime for the one-point functions of a scalar field (Wick square, mixed stress tensor, charged current), on a declared smooth completion with Ward prescription. Version 1 classified the contact freedom for the Friedrichs state. Version 2 extended this to non-Friedrichs, thermal and massive states, and proved that no canonical dilation-invariant point exists. *Version 3* adds six results. (1) For general non-diagonal $U(2)$ extensions with $H \geq 0$, the leading apex coefficients are those of the ultraviolet Lagrangian of H. The off-diagonal sector has closed-form leading asymptotics $\propto \kappa_{01}(H)\, r^{-1/\alpha}$, with an angle-dependent stress tensor and a radial current $J_r \neq 0$; it has a unique Ward-compatible extension with no residue and no contact. (2) Explicit flower-pot cores show that the zero-thickness limit of the apex contact is not universal and is decided by the curvature coupling. For $\xi = 0$ the limit is finite and model-dependent. For $\xi > 0$ the contact diverges like $-\frac{1}{4\pi}\log\log(1/R)$. For $\xi < 0$ there is no vacuum. The $(\xi - \frac{1}{6})R\,\delta$ term moves the contact off the dilation orbit. (3) The restriction $\varphi|_{\mathscr{W}}$ exists for $\beta = 0$ (a dimension-1 generalized free field on $\mathscr{W}$) and vanishes for $\beta \neq 0$. Boundary values a_j of deficient modes are worldsheet generalized free fields of dimension $1 - \nu_j$; for E_λ they flow to dimension $1 + \nu_j$. The normal-ordered

Wick square smears on $\mathscr{W}$, but its covariant c-number part does not. (4) For the massless Dirac field with flux we give closed forms for $T^{\mu}{}_{\nu}$ and J^{φ} ($\langle\bar{\psi}\psi\rangle = 0$), the residues, and the two-parameter contact space. These agree with Linet and with Bezerra de Mello–Saharian. A disagreement with Spinelly–Bezerra de Mello is disclosed. (5) The first-order $\lambda\varphi^4$ tadpole turns the free contact c into a physical r^{-2} term; beyond first order there is a precise obstruction. (6) A re-audit of version 2 confirms its theorems and corrects the physical interpretation of its core theorem. All checks are symbolic or numerical, and no formal certificate is asserted.

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Book expansion: object map and inherited vs re-proved

toprule Layer Role in this sheet

midrule Smooth host Prescribed conical product (I-B)

Extension Friedrichs / non-Friedrichs / $U(2)$ data

State Ground / thermal / massive as labelled

Regular one-points Wick square, mixed stress, charged current off $\mathscr{W}$

Distributional completion Contact space across the apex worldsheet

bottomrule **Reading split.** Sections that re-prove statements are marked proved here; formulae taken from earlier companion versions are supplied input. Historical audit and programme-ledger material remain documentary and are not re-executed. Version labels in this book edition are normalized to **v4**; do not cite this chapter as ``version 1" or ``version 3" in sibling bibliographies.

Version note (version 3)

This version supersedes version 2 and is self-contained. The version-2 text is retained except where Section corrects it. New are Sections –, the updated ledger, and the references marked below. Reference markers: ^* read in full for the cited equation; ^‡ abstract or a quoted passage verified online; ^† cited from standard knowledge, not re-verified. Scripts are code/v3_.py, and logs are logs/v3_.json. Nothing in this version depends on unpublished results beyond the I-B-VP series.

Version note (version 2)

What was audited. Version 1 was written without a compiler run, numerical execution or formal verification. For version 2 every theorem, lemma and displayed computation of version 1 was rechecked. The tools were exact symbolic algebra (sympy) and high-precision quadrature with off-centre, non-symmetric test functions (mpmath, 30 digits). The checks are collected in Section . All regular coefficients agree with the independent companion

derivation I-B-VP-AB [31], with exact residual 0. Every conversion, raw defect, repair coefficient, residue in and contact dimension of Theorem is confirmed. *No mathematical error was found in the core results.*

Corrections and sharpenings (E1–E7).

E1 *Source did not compile.* The macro verb | D| used verb || without mathrsfs, and verb | ket |, verb | bra | and verb | proj | were undefined. pdflatex stopped with errors. Fixed.

E2 *Unnumbered labels.* Labels eq:delta and eq:K sat in unnumbered displays. Fixed.

E3 *Redundant hypotheses in Theorem* . Exact linear algebra on the contact module (Section , Table) shows that the following already give dimension three:

• the scaling-degree bounds;

• longitudinal translation and boost invariance;

• $\bar{g}$-symmetry;

• the Ward identities. Transverse rotation covariance, meridian reflection and almost homogeneity are not needed. Conservation alone excludes the radial current contact and the parity-odd tensor. Charge conjugation is needed only for $w = 0$ at self-conjugate sectors. The theorem is stated as in version 1, and Proposition adds the reduced form.

E4 *Positivity sharpened.* For every $s \neq 0$, of either sign, no extension of F° is a signed Radon measure (Section).

E5 *Non-homogeneity for every extension.* Version 1 proved non-homogeneity only for extensions of the form $U_L + c\delta_{\mathscr{W}}$. In fact no extension of sr^{-2}, $s \neq 0$, of any order is transversally homogeneous (Proposition).

E6 *An inaccurate remark.* Version 1 said almost homogeneity is what excludes $L^{-2}\delta_{\mathscr{W}}$ from the longitudinal stress. In the massless class the trace identity excludes it anyway, as Appendix of version 1 itself notes. The remark is corrected. With mass, a contact $m^2\delta_{\mathscr{W}}$ is forced, tied to c (Theorem (v)).

E7 *Scope of the Ward representative.* Definition agrees with $\nabla^g T = 0$ only for regular data whose $\bar{g}$-lowered stress

is symmetric, for instance diagonal data. For the non-diagonal $\beta = \frac{1}{2}$ states it fails by an explicit amount (Theorem). Version 2 adds the Killing–homothety representative (Definition), which agrees with $\nabla^g T = 0$ for all stationary boost-invariant data.

Companion records. The regular inputs are the Friedrichs coefficients of I-B-VP [9] and I-B-VP-AB [31], and the non-Friedrichs closed forms of I-B-VP-SAE [32].

• The theorem labels cited in version 1 (thm:phi2, thm:T-real, thm:T-complex) are those of a supplied I-B-VP-v2 source. In the versions available to us the same formulas appear as I-B-VP-AB Theorems thm:phi2, thm:T and thm:J. They agree exactly.

• I-B-VP-0 [10] was not available to us in source form. Its current coefficient is used only as a cited input. It coincides with I-B-VP-AB Theorem thm:J (residual 0).

What is new.

• Sections – are new: the potential theorem, the non-Friedrichs, thermal and massive extensions, the renormalization no-go and the field algebra.

• Section records the audit.

• The ledger (Section) is updated. Literature is marked ^* (read in full or in the quoted part), ^‡ (bibliographic data and abstract confirmed online) or ^† (from memory, not re-verified).

Purpose, inputs, and the essential qualification

Cheung's I-B prescribes a regular conical spacetime and a separate finite operational interface [7]. I-B-G constructs intrinsic scalar Green operators after a spatial extension has been supplied [8]. I-B-VP-v2 constructs the distinguished massless Friedrichs spectral ground state, proves its local Hadamard form, and evaluates its Wick square and stress tensor [9]. I-B-VP-0-v2 evaluates the charged circulating current [10]; its coefficient is used here only as a cited input and agrees exactly with the independent derivation of I-B-VP-AB [31]. The non-Friedrichs closed forms used in version 2 are those of I-B-VP-SAE [32].

Those observables are smooth on the regular spacetime. Their singular radial behaviours do not, without an additional prescription, define distributions across the completed apex worldsheet. The present paper supplies one such prescription and classifies its finite freedom.

The qualification that cannot be omitted. The metric completion is not a smooth Lorentzian manifold carrying the displayed conical metric at its apex. Before speaking of tensor distributions there, one must specify the test space, smooth structure, tensor bundle, density, and meaning of the identities to be preserved. In particular, an expression such as $\Gamma(g)T$, with an apex-supported T, is not automatically a product of distributions.

We therefore construct distributions on a declared smooth completion and preserve a declared smooth representative of the regular invariant Ward equations. We do *not* claim that this defines the Levi-Civita derivative of arbitrary tensor distributions for the singular metric. The distinction is consistent with the general cautions about concentrated sources and distributional geometry in Geroch–Traschen [5].

Enter and the non-extraction boundary. The philosophy of Enter distinguishes a destructive readout from a quantum-return map, a record from a decoder, and an analogy from an identification [6]. I-B and En09-A implement a finite destructive instrument returning a classical code and a surviving residual, not the measured clock register [7, 19]. That instrument does not supply the field, apex condition, state, continuum test space, or renormalization prescription used below.

The cone, worldsheet, Friedrichs extension, spectral state, and completion prescription remain declared inputs. No continuum distribution is extracted from a finite conditional density matrix. Conservation does not impose Einstein dynamics [7, prop:not-source] [11, prop:noeinstein].

Analytical context and contribution. Extension by Taylor subtraction, scaling degree, and distributions supported on a submanifold are classical tools [1, 2, 3]. Almost-homogeneous extensions and their dilation residues are likewise established concepts [4]. Our contribution is their explicit application to the simultaneous one-point data, including the stress Ward repair and the complete contact space in the prescription fixed here.

VII-D's strong finite-part construction and VII-H's explicitly restricted counterterm space are methodological

comparisons [13, 14]. Their numerical kernels and contact classes are not imported. Similarly, the relative-lifting and normal-jet distinctions in XV-B motivate a compatibility check, not an identification with its characteristic graph [15].

Proof dependencies. The displayed regular coefficients in Section are the input identified with the Friedrichs state by the cited companions. Every assertion about their distributional extension, cutoff limit, Ward repair, scaling, and contact classification is proved below. No missing interacting product or characteristic composition is used.

The completion, regular data, and Ward prescription

Geometry and the additional smooth structure

Fix

$$0 < \alpha < 1, \qquad \mu = 2\pi(1-\alpha), \qquad 0 \le \beta < 1, \qquad \xi \in \mathbb{R}.$$

The regular host is

$$M = \mathbb{R}_t \times (0,\infty)_r \times (\mathbb{R}/2\pi\mathbb{Z})_\varphi \times \mathbb{R}_z,$$

$$g = -\mathrm{d}t^2 + \mathrm{d}r^2 + \alpha^2 r^2 \mathrm{d}\varphi^2 + \mathrm{d}z^2.$$

Its metric completion adds

$$\mathscr{W} = \mathbb{R}_t \times \{o\} \times \mathbb{R}_z,$$

a worldsheet, not a worldline [7, 8].

Assumption (Smooth completion and density). Identify the topological completion with

$$\overline{M} = \mathbb{R}^{1+1}_y \times \mathbb{R}^2_x, \qquad y = (t,z), \qquad x = (x^1,x^2) = (r\cos\varphi, r\sin\varphi).$$

Equip it with this Cartesian smooth structure and the auxiliary smooth metric

$$\bar{g} = -\mathrm{d}t^2 + \mathrm{d}z^2 + (\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2.$$

The pairing density is

$$\mathrm{d}\Omega = \alpha\, \mathrm{d}t\, \mathrm{d}z\, \mathrm{d}^2x.$$

It equals the host volume density on M.

All tensor distributions below are sections of the ordinary smooth tensor bundles of $\overline{M}$. Stress is extended as a mixed tensor $T^{\mu}{}_{\nu}$; the current is extended first as a contravariant vector J^{μ}.

These x-coordinates are *not* the developing coordinates $(r\cos(\alpha\varphi), r\sin(\alpha\varphi))$. Assumption is additional data. For a quotient angle $\alpha = 1/N$, it is not automatically the orbifold test-function structure of the covering plane.

Write

$$\delta_{\mathscr{W}} : \quad \langle \delta_{\mathscr{W}}, f \rangle = \int_{\mathbb{R}^{1+1}} f(y, 0)\, \mathrm{d}t\, \mathrm{d}z.$$

Thus, as a generalized function relative to $\mathrm{d}\Omega$, $\delta_{\mathscr{W}} = \alpha^{-1}\delta^{(2)}_{\mathrm{Leb}}(x)$. Derivatives are transposed with respect to the constant density . Put

$$\Delta = \partial_1^2 + \partial_2^2, \qquad E = x^i \partial_i, \qquad \sigma_\alpha = 2\pi\alpha,$$

$$K_{ij} = \partial_i \partial_j - \delta_{ij}\Delta.$$

Indices A, B range over t, z, and i, j over $1, 2$.

Let ε be the positive quarter-turn matrix,

$$\varepsilon = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad R^i = \varepsilon^i{}_j x^j.$$

The smooth vector R is ∂_φ on M.

Regular one-point data

The charged field obeys

$$\Phi(\varphi + 2\pi) = e^{2\pi i\beta}\Phi(\varphi)$$

in a lifted parallel frame. Its spatial extension and state are the massless Friedrichs operator and distinguished spectral ground state of I-B-VP-v2 [9, def:state, lem:infrared, thm:hadamard]. For the real field take $\beta = 0$.

Introduce a multiplicity

$$n = \begin{cases} 1, & \text{one real neutral scalar,} \\ 2, & \text{one charged complex scalar.} \end{cases}$$

Define

$$P_\beta = \alpha^{-2}(1 - 6\beta(1-\beta)) - 1,$$

$$C_\beta = \alpha^{-4}(1 - 30\beta^2(1-\beta)^2) - 1,$$

$$s = \frac{P_\beta}{48\pi^2}, \qquad h = \frac{nC_\beta}{1440\pi^2}, \qquad q = \xi - \frac{1}{6},$$

$$k = nqs, \qquad a = h + 4k, \qquad b = h - 2k.$$

The regular Wick square is

$$F^\circ = \frac{s}{r^2},$$

meaning $\langle\varphi^2\rangle_{\mathrm{ren}}$ for the real field and $\langle\Phi^\dagger\Phi\rangle_{\mathrm{ren}}$ for the complex field. The effective field square in the stress trace is nF°.

In the smooth coordinates of Assumption , the regular mixed stress is

$$(T^\circ)^A{}_B = \frac{a}{r^4}\delta^A_B, \qquad (T^\circ)^A{}_i = (T^\circ)^i{}_A = 0,$$

$$(T^\circ)^i{}_j = \frac{b}{r^4}\left(4\frac{x^i x_j}{r^2} - 3\delta^i_j\right).$$

These are precisely the mixed components $r^{-4}\mathrm{diag}\,(a, b, -3b, a)$ in order (t, r, φ, z) [9, thm:phi2, thm:T-real, thm:T-complex].

For the charged field put

$$j = \frac{\beta(1-\beta)(2\beta-1)}{12\pi^{2}\alpha^{2}}.$$

With

$$J_{\mu} = \mathrm{i}\big(\Phi^{\dagger}D_{\mu}\Phi - (D_{\mu}\Phi)^{\dagger}\Phi\big),$$

I-B-VP-0-v2 gives

$$(J^{\circ})^{i} = \frac{j\,\varepsilon^{i}{}_{j}x^{j}}{\alpha^{2}\,r^{4}}, \qquad (J^{\circ})^{A} = 0, \qquad J^{\circ}_{\varphi} = \frac{j}{r^{2}}.$$

The current uses the complex first jet of the regular covariance, not the real part used for stress [10, def:current, thm:current]. Set $j = 0$ for the real field.

Which Ward identities are extended?

Definition (Prescribed Ward representative). For the stationary distributions considered here, use

$$\mathfrak{D}_{\nu}T := \partial_{\mu}T^{\mu}{}_{\nu}, \qquad \mathfrak{d}J := \partial_{\mu}J^{\mu}, \qquad \mathfrak{t}T := T^{\mu}{}_{\mu}.$$

For a stationary scalar distribution G, define the smooth reference improvement

$$\mathscr{I}[G]^{\mu}{}_{\nu} = \delta^{\mu}_{\nu}\mathrm{Box}_{\bar{g}}G - \bar{\partial}^{\mu}\partial_{\nu}G.$$

In the present translation-invariant sector,

$$\mathscr{I}[G]^{A}{}_{B} = \delta^{A}_{B}\Delta G, \qquad \mathscr{I}[G]^{i}{}_{j} = -K_{ij}G, \qquad \mathscr{I}[G]^{A}{}_{i} = \mathscr{I}[G]^{i}{}_{A} = 0.$$

Lemma (Agreement with the regular invariant equations). *For the regular tensors ,*

$$\nabla^{g}_{\mu}(T^{\circ})^{\mu}{}_{\nu} = \mathfrak{D}_{\nu}T^{\circ} = 0, \qquad \nabla^{g}_{\mu}(J^{\circ})^{\mu} = \mathfrak{d}J^{\circ} = 0.$$

For every smooth radial scalar $G(r)$, the mixed improvement $(g\mathrm{Box}_{g} - \nabla^{g}\nabla^{g})^{\mu}{}_{\nu}G$ equals $\mathscr{I}[G]^{\mu}{}_{\nu}$. Consequently,

$$\mathfrak{t}T^{\circ} = 3q\,\Delta(nF^{\circ}),$$

$$T^{\circ}(\xi) - T^{\circ}(1/6) = nq\,\mathscr{I}[F^{\circ}].$$

Proof. In polar coordinates the only nonzero difference between the transverse Christoffel symbols of g and $\bar{g}$ is

$$\Gamma(g)^{r}_{\varphi\varphi} - \Gamma(\bar{g})^{r}_{\varphi\varphi} = (1-\alpha^{2})r.$$

Its contraction with the diagonal mixed tensor in vanishes in the mixed divergence. The two volume densities differ by the constant α, so vector divergences agree as well.

For a radial scalar, both mixed Hessians have radial entry G'' and angular entry G'/r; both wave operators give $G'' + G'/r$. This proves the improvement assertion. Finally,

$$a - b = 6k, \qquad \Delta r^{-2} = 4r^{-4}$$

on $r > 0$, which gives the trace and improvement identities.

Remark (Not an unrestricted singular-metric derivative). Definition extends a regular, already-cancelled invariant equation by a smooth differential operator. It does not define products of the conical connection with delta derivatives. Likewise, the trace is the smooth mixed trace; we do not first lower or raise distributional indices with the singular metric at $\mathscr{W}$. All claims of Ward compatibility below refer to this definition, unless the Killing–homothety form of Definition is stated (version 2).

Killing–homothety Ward identities (new in version 2)

Definition uses the $\bar{g}$-divergence of the mixed tensor. By Lemma , this equals $\nabla^{g}T$ for diagonal data, because the Christoffel difference $\Gamma(g)^{r}_{\varphi\varphi} - \Gamma(\bar{g})^{r}_{\varphi\varphi}$ multiplies $T^{\varphi}{}_{r}$, which then vanishes. In general,

$$\partial_{i}T^{i}{}_{j} = \nabla^{g}_{\mu}T^{\mu}{}_{j} + (1-\alpha^{2})\,\omega^{\perp}_{j}\,T^{\varphi}{}_{r} \qquad (r > 0),$$

with $\omega^{\perp} = (-\sin\varphi, \cos\varphi)$ and $T^{\varphi}{}_{r}$ the polar mixed component (checked symbolically, Section). The smooth vector fields

$$R^{i} = \varepsilon^{i}{}_{j}x^{j}, \qquad X^{i} = x^{i}$$

are a Killing field and a transverse homothety of g: $\mathscr{L}_R g = 0$, and $\mathscr{L}_X g_{ij} = 2g_{ij}$ on the transverse block. On M the density $\alpha\, \mathrm{d}^2x$ equals $\sqrt{g}\, \mathrm{d}^2x$ exactly, so the g-divergence of a vector field is its coordinate divergence.

Definition (Killing–homothety representative). For stationary, z-translation-invariant tensor distributions $T^{\mu}{}_{\nu}$ on $\overline{M}$, put

$$\mathrm{W1}_A(T) = \partial_\mu T^{\mu}{}_{A}, \qquad \mathrm{W2}(T) = \partial_i(T^{i}{}_{j} R^{j}),$$

$$\mathrm{W3}(T) = \partial_i(T^{i}{}_{j} X^{j}) - T^{i}{}_{i}.$$

All three use only smooth multipliers and ordinary distributional derivatives.

Lemma. *For smooth regular data on M with $T_{\mu\nu}$ g-symmetric,*

$$\nabla^{g}_{\mu} T^{\mu}{}_{\nu} = 0 \quad \Leftrightarrow \quad \mathrm{W1} = \mathrm{W2} = \mathrm{W3} = 0 \text{ on} M.$$

For data whose $\bar{g}$-lowered stress is symmetric, $\mathrm{W1} = \mathrm{W2} = \mathrm{W3} = 0$ *is implied by* $\mathfrak{D}T = 0$. *On M the converse holds for every g-symmetric tensor (first statement); at the apex the two representatives are compared in Proposition and Table .*

Proof. For a Killing or homothetic field ξ, $\nabla_\mu(T^{\mu}{}_{\nu}\xi^{\nu}) = (\nabla_\mu T^{\mu}{}_{\nu})\xi^{\nu} + T^{\mu\nu}\nabla_{(\mu}\xi_{\nu)}$. The last term is 0 for R and $T^{i}{}_{i}$ for X. Since R and X span the transverse tangent space at $r > 0$, the transverse components of ∇T vanish iff W2 and W3 vanish. The longitudinal components are W1. For the second statement, expand $\partial_i(T^{i}{}_{j}\xi^{j}) = (\partial_i T^{i}{}_{j})\xi^{j} + T^{i}{}_{j}\partial_i\xi^{j}$. Here $\partial_i R^{j} = \varepsilon^{j}{}_{i}$ contracts to zero against a $\bar{g}$-symmetric tensor, and $\partial_i X^{j} = \delta^{j}_{i}$.

Strong radial finite parts

The scalar generating distribution

Fix a length $L > 0$ and a smooth profile

$$\rho : [0,\infty) \to [0,1], \qquad \rho(u) = 0 \ (u \le 1), \qquad \rho(u) = 1 \ (u \ge 2).$$

Put

$$c_\rho = \int_1^2 \frac{\rho(u) - 1}{u}\,\mathrm{d}u.$$

Definition (Generating finite part). Define

$$U^\rho_{L,\varepsilon} = \rho(r/\varepsilon) r^{-2} + \sigma_\alpha \Big(\log \frac{\varepsilon}{L} - c_\rho \Big) \delta_{\mathscr{M}}.$$

Its limit will be denoted by U_L. The first term is interpreted as a locally integrable function with the density $\mathrm{d}\Omega$.

Theorem (Existence, profile independence, and strong convergence). *The limit*

$$U_L = \lim_{\varepsilon \mathrm{downarrow}\, 0} U^\rho_{L,\varepsilon}$$

exists strongly in $\mathcal{D}'(\overline{M})$, is independent of ρ, and restricts to r^{-2} on M. It is rotation invariant, translation invariant in t, z, and real.

For every compact test support there is a constant C such that

$$\left| \mathrm{langle} U_{L,\varepsilon}^\rho - U_L, f \right\rangle| \le C \varepsilon^2 p_2(f),$$

where p_2 is a fixed-support smooth seminorm. In particular, all fixed distributional derivatives converge strongly, with the corresponding extra derivatives in the seminorm.

Proof. Let

$$\bar{f}(y, r) = \frac{1}{2\pi} \int_0^{2\pi} f(y, r\cos\varphi, r\sin\varphi)\,\mathrm{d}\varphi.$$

Taylor's theorem and angular averaging give

$$\bar{f}(y, r) = f(y, 0) + O(r^2 p_2(f))$$

uniformly on the relevant compact set.

For the sharp cutoff, the expression

$$\int_{r > \varepsilon} r^{-2} f\,\mathrm{d}\Omega + \sigma_\alpha \log(\varepsilon/L) \int f(y, 0)\,\mathrm{d}^2 y$$

has a limit: subtract $f(y,0)$ in a fixed small radial neighbourhood and integrate the remainder with measure $\sigma_\alpha\,\mathrm{d}r/r$. The omitted remainder on $0 < r < \varepsilon$ is $O(\varepsilon^2)$.

The smooth cutoff differs from the sharp cutoff only on $\varepsilon < r < 2\varepsilon$. Its constant Taylor term contributes

$$\sigma_\alpha c_\rho \int f(y,0)\,\mathrm{d}^2 y,$$

which is exactly removed in . Its remaining contribution is $O(\varepsilon^2 p_2(f))$. This proves existence, profile independence, and .

The estimates are uniform on bounded sets of tests with fixed compact support, and hence give strong distributional convergence. The symmetries and reality follow directly from the construction. Derivative convergence follows by transposition.

Direct componentwise finite parts

The following elementary extension lemma will also be used to audit direct cutoff prescriptions.

For $p \in \{2,3,4\}$ and a smooth angular coefficient A, define

$$f_A^\circ(x) = r^{-p} A(x/r).$$

For a test f, put

$$M_k^A[f] = \frac{\alpha}{k!} \int_{\mathbb{R}^{1+1}} \int_0^{2\pi} A(\omega)(\omega\cdot\partial_x)^k f(y,0)\,\mathrm{d}\varphi\,\mathrm{d}^2 y, \qquad \omega = (\cos\varphi, \sin\varphi).$$

When $k < p-2$, set

$$m_{p,k}^\rho = \int_0^\infty \rho(u) u^{k+1-p}\,\mathrm{d}u.$$

Lemma (Finite-part formula for the component kernels). *The limit*

$$\left\langle \mathscr{R}_L[f_A^\circ], f \right\rangle = \lim_{\varepsilon \,\text{downarrow}\, 0} \{ \int \rho(r/\varepsilon) f_A^\circ f\,\mathrm{d}\Omega - \sum_{k=0}^{p-3} \varepsilon^{k+2-p} m_{p,k}^\rho M_k^A[f]$$

$$+\Big(\log\frac{\varepsilon}{L}-c_\rho\Big)M^A_{p-2}[f]\}$$

exists and is independent of the profile. The sum is empty when p = 2. It defines a distribution of transverse scaling degree at most p, equal to f°_A *off* $\mathscr{W}$.

For the scalar, stress, and vector kernels below, parity improves the fixed-support convergence remainder to $O(\varepsilon^2)$.

Proof. Expand the test transversely through degree $p-2$. Radial integration of the k-th term gives the power ε^{k+2-p} when $k < p-2$, and a logarithm when $k = p-2$. These are precisely the subtractions in . The Taylor remainder is integrable.

Comparing two profiles on their transition annuli gives the same moment contributions, so the normalized limit is profile independent. Taylor remainder estimates prove continuity and the stated scaling-degree bound.

For r^{-2}, angular odd terms vanish. For the stress kernels the angular coefficient is even, so the cubic Taylor term vanishes. For x_i/r^4, the angular coefficient is odd, so the quadratic Taylor term vanishes. In these cases the first unremoved angular term yields an $O(\varepsilon^2)$ remainder, bounded respectively by test seminorms of orders $2, 4, 3$.

Remark (The actual divergence powers). The codimension is two. Thus r^{-4} produces an ε^{-2} divergence and a logarithmic second-jet subtraction, not an ε^{-4} divergence. For the circulating vector, the prospective ε^{-1} term has zero angular moment; only a logarithmic first-jet subtraction remains.

We shall write

$$V^{\mathrm{c}}_L=\mathscr{R}_L[r^{-4}],\qquad Q^{\mathrm{c}}_{L,i}=\mathscr{R}_L[x_i/r^4],$$

$$S^{\mathrm{c}}_{L,ij}=\mathscr{R}_L\Big[r^{-4}\Big(4\frac{x_ix_j}{r^2}-3\delta_{ij}\Big)\Big].$$

The superscript c denotes componentwise cutoff finite parts, not a conserved prescription.

Simultaneous extensions and the exact Ward repair

A derivative-compatible prescription

Define

$$\widehat{F}_L = sU_L,$$

$$(\widehat{T}_L)^A{}_B = \frac{a}{4}\delta^A_B \Delta U_L,\ (\widehat{T}_L)^A{}_i = (\widehat{T}_L)^i{}_A = 0,$$

$$(\widehat{T}_L)^i{}_j = \frac{b}{2}K_{ij}U_L,$$

and

$$(\widehat{J}_L)^i = -\frac{j}{2\alpha^2}\varepsilon^i{}_k\partial_k U_L, \qquad (\widehat{J}_L)^A = 0.$$

Theorem (Simultaneous Ward-compatible extension). *The distributions – are real extensions of the regular data –. They have transverse scaling degrees at most* $2, 4, 3$*, respectively. They satisfy*

$$\mathfrak{D}_\nu \widehat{T}_L = 0, \qquad \mathfrak{d}\widehat{J}_L = 0,$$

$$\mathfrak{t}\widehat{T}_L = 3q\,\Delta(n\widehat{F}_L),$$

$$\widehat{T}_L(\xi) - \widehat{T}_L(1/6) = nq\,\mathscr{I}[\widehat{F}_L].$$

The reference-lowered stress $\bar{g}_{\lambda\mu}(\widehat{T}_L)^\mu{}_\nu$ *is symmetric.*

Replacing U_L *by* $U^\rho_{L,\varepsilon}$ *gives strongly convergent regulated extensions. The errors are bounded by* $C\varepsilon^2 p_2$*,* $C\varepsilon^2 p_4$*, and* $C\varepsilon^2 p_3$ *for field square, stress, and current.*

Proof. On $r > 0$,

$$\frac{1}{4}\Delta r^{-2} = r^{-4}, \qquad -\frac{1}{2}\partial_i r^{-2} = \frac{x_i}{r^4},$$

$$\frac{1}{2}K_{ij}r^{-2} = r^{-4}\left(4\frac{x_i x_j}{r^2} - 3\delta_{ij}\right).$$

This proves exact regular-part matching.

All derivatives in – are derivatives of a defined distribution. Since

$$\partial_i K_{ij} = 0, \qquad \partial_i \varepsilon^i{}_k \partial_k = 0,$$

conservation follows without a boundary-flux omission. The mixed trace is

$$\mathrm{t}\widehat{T}_L = \left(\frac{a}{2} - \frac{b}{2}\right)\Delta U_L = 3k\Delta U_L = 3q\,\Delta(n\widehat{F}_L).$$

The changes of $a/4$ and $b/2$ under a change from conformal coupling are nqs and $-nqs$, respectively, giving . Symmetry is immediate from the block expressions. The scaling-degree and strong-convergence claims follow from Theorem .

This is a construction of c-number one-point distributions. It is not an extension of the two-point function, quantum state, or field algebra to $\mathscr{W}$.

The direct cutoff prescription need not obey the Ward equations

Proposition (Exact finite conversions). *The direct finite parts satisfy*

$$\frac{1}{4}\Delta U_L = V_L^{\mathrm{c}} - \frac{\sigma_\alpha}{4}\Delta\delta_{\mathscr{W}},$$

$$-\frac{1}{2}\partial_i U_L = Q_{L,i}^{\mathrm{c}} + \frac{\sigma_\alpha}{4}\partial_i\delta_{\mathscr{W}},$$

$$\frac{1}{2}K_{ij}U_L = S_{L,ij}^{\mathrm{c}} + \sigma_\alpha\left(\frac{7}{16}\delta_{ij}\Delta - \frac{3}{8}\partial_i\partial_j\right)\delta_{\mathscr{W}}.$$

In particular,

$$\partial_i S_{L,ij}^{\mathrm{c}} = -\frac{\sigma_\alpha}{16}\partial_j\Delta\delta_{\mathscr{W}}, \qquad S_{L,ii}^{\mathrm{c}} = -2V_L^{\mathrm{c}}.$$

Proof. Profile independence permits the sharp cutoff in this comparison. Twice integrating by parts on $r > \varepsilon$, and

expanding the boundary test jets, gives

$$\partial_i U_L = \mathscr{R}_L[\partial_i r^{-2}] - \frac{\sigma_\alpha}{2}\partial_i \delta_{\mathscr{W}},$$

$$\partial_i\partial_j U_L = \mathscr{R}_L[\partial_i\partial_j r^{-2}] - \frac{3\sigma_\alpha}{4}\partial_i\partial_j \delta_{\mathscr{W}} - \frac{\sigma_\alpha}{8}\delta_{ij}\Delta\delta_{\mathscr{W}}.$$

The boundary computations are detailed in Appendix . Their trace gives ; the first identity gives ; subtracting the traced second identity gives . Taking a divergence of proves the first identity in . The second follows either by tracing the regular angular kernel before applying the linear finite-part operation, or directly from the moment subtractions.

Define the preliminary componentwise lift by

$$(T_L^{\mathrm{c}})^A{}_B = a\delta^A_B V_L^{\mathrm{c}}, \qquad (T_L^{\mathrm{c}})^i{}_j = bS^{\mathrm{c}}_{L,ij}, \qquad (J_L^{\mathrm{c}})^i = \frac{j}{\alpha^2}\varepsilon^i{}_k Q^{\mathrm{c}}_{L,k}.$$

Corollary (Computed defects and an admissible correction). *The preliminary lift has*

$$\mathfrak{D}_j T_L^{\mathrm{c}} = -\frac{b\sigma_\alpha}{16}\partial_j\Delta\delta_{\mathscr{W}},$$

$$\mathfrak{t} T_L^{\mathrm{c}} - 3q\Delta(\widehat{nF}_L) = 3k\sigma_\alpha\Delta\delta_{\mathscr{W}},$$

$$\mathfrak{d} J_L^{\mathrm{c}} = 0.$$

The corrections

$$(\widehat{T}_L - T_L^{\mathrm{c}})^A{}_B = -\frac{a\sigma_\alpha}{4}\delta^A_B\Delta\delta_{\mathscr{W}},$$

$$(\widehat{T}_L - T_L^{\mathrm{c}})^i{}_j = b\sigma_\alpha\Big(\frac{7}{16}\delta^i_j\Delta - \frac{3}{8}\partial^i\partial_j\Big)\delta_{\mathscr{W}},$$

$$(\widehat{J}_L - J_L^{\mathrm{c}})^i = \frac{j\sigma_\alpha}{4\alpha^2}\varepsilon^i{}_k\partial_k\delta_{\mathscr{W}}$$

give exactly Theorem .

Proof. The divergence follows from . The raw trace is

$$2(a-b)V_L^c = 12kV_L^c.$$

Equation makes $3q\Delta(n\widehat{F}_L) = 12kV_L^c - 3k\sigma_\alpha \Delta\delta_{\mathscr{W}}$, proving . The vector in differs from a gradient by a delta derivative, so its rotation by ε is conserved. The corrections are the three conversions multiplied by their regular coefficients.

Remark (Subtraction, not algebra). The raw finite part exists, but is not automatically a Ward-compatible extension. Corollary evaluates the defect and supplies an actual correction in an explicit supported space. This is the relevant compatibility calculation; merely naming a relative-lifting criterion would not supply it [13, 15, 16, 17]. Nothing here defines a product of singular observables.

The complete contact space in the declared class

A finite-dimensional ambiguity requires more than a bound on normal derivative order. We now fix the full class.

Assumption (Admissible one-point extensions). For each fixed (α, β, ξ), consider real distributional extensions (F, T, J) of the displayed regular data satisfying:

1. Translation invariance along $\mathscr{W}$, covariance under longitudinal Lorentz transformations, and transverse rotations. The reference-lowered stress is symmetric.

2. Charge conjugation sends β to $1-\beta$: F, T are even and J is odd. Transverse reflection reverses the meridian and therefore also sends β to $1-\beta$. These requirements apply to the family of sectors.

3. The componentwise transverse scaling degrees are at most $2, 4, 3$ for F, T, J^i, respectively, and the extensions are almost homogeneous of logarithmic order at most one:

$$(E+2)^2 F = 0, \qquad (E+4)^2 T^\mu{}_\nu = 0, \qquad (E+3)^2 J^\mu = 0.$$

These are equations for the stated Cartesian component distributions.

4. The Ward representative is Definition , and the required identities are

$$\mathfrak{D}T = 0, \qquad \mathfrak{d}J = 0, \qquad \mathfrak{t}T = 3q\,\Delta(nF).$$

For a coupling-dependent family, F, J are required to be independent of ξ, and the improvement identity is required with this same F.

(Corrected in version 2, E6.) A new power-law defect such as $L^{-2}\delta_{\mathscr{W}}$ in the longitudinal stress is excluded by the trace identity itself, and also by almost homogeneity; see Proposition . It does not require strict homogeneity, which will generally be impossible. The class is a prescription, not a classification of every possible singular-background renormalization.

Define the conserved, traceless mixed contact tensor

$$\mathscr{H}^{A}{}_{B} = \delta^{A}_{B}\Delta\delta_{\mathscr{W}},\ \mathscr{H}^{i}{}_{j} = 2K_{ij}\delta_{\mathscr{W}},\ \mathscr{H}^{A}{}_{i} = \mathscr{H}^{i}{}_{A} = 0.$$

Indeed, $\mathfrak{D}\mathscr{H} = 0$ and $\mathfrak{t}\mathscr{H} = 2\Delta\delta_{\mathscr{W}} - 2\Delta\delta_{\mathscr{W}} = 0$.

Theorem (Complete ambiguity and its dimension). *Every extension in Assumption is uniquely of the form*

$$F = \widehat{F}_{L} + c\,\delta_{\mathscr{W}},$$

$$T^{\mu}{}_{\nu} = (\widehat{T}_{L})^{\mu}{}_{\nu} + \tau\mathscr{H}^{\mu}{}_{\nu} + nq\,c\,\mathscr{I}[\delta_{\mathscr{W}}]^{\mu}{}_{\nu},$$

$$J^{i} = (\widehat{J}_{L})^{i} + w\,\varepsilon^{i}{}_{k}\partial_{k}\delta_{\mathscr{W}}, \qquad J^{A} = 0,$$

with real c, τ, w.

Conversely, these formulas give all the required extensions. For coupling-dependent families, c, τ, w can be taken independent of ξ, and then the improvement identity holds.

The coefficients c, τ are even in β, and w is odd. Consequently:

1. *for a generic charged sector $\beta \notin \{0, 1/2\}$, the contact space has dimension* 3*;*

2. *at a self-conjugate charged sector $\beta = 0$ or $1/2$, $w = 0$, and the dimension is* 2*;*

3. *for the real neutral field, there is no charged current and the dimension is* 2*. The dimension is for fixed*

background and field parameters, not for arbitrary functions of those parameters.

Proof. The difference of two extensions is supported on $\mathscr{W}$. The structure theorem for distributions supported on a submanifold expresses it as a finite sum of transverse delta derivatives with coefficient distributions on $\mathscr{W}$ [2, Section 8] [3, Theorem 2.3.5]. Translation invariance forces those coefficient distributions to be constants times the longitudinal Lebesgue distribution.

A transverse derivative of order m obeys

$$E(\partial^{\gamma}\delta_{\mathscr{W}}) = -(2 + m)\partial^{\gamma}\delta_{\mathscr{W}}, \qquad |\gamma| = m.$$

The scaling-degree bounds and therefore allow only order 0 for the field-square difference, order 2 for stress, and order 1 for current. For example, a stress term proportional to $\delta_{\mathscr{W}}$ has $(E + 4)^2\delta_{\mathscr{W}} = 4\delta_{\mathscr{W}}$, and is excluded.

Rotation covariance and the reflection/charge-conjugation conditions give

$$\Delta F = c\delta_{\mathscr{W}},$$

$$\Delta T^{A}{}_{B} = u\,\delta^{A}_{B}\Delta\delta_{\mathscr{W}}, \qquad \Delta T^{i}{}_{j} = v\,\delta^{i}_{j}\Delta\delta_{\mathscr{W}} + z\,\partial^{i}\partial_{j}\delta_{\mathscr{W}},$$

with no mixed blocks. The parity-odd symmetric transverse tensor is excluded by the even charge-conjugation behaviour of stress combined with meridian reflection.

For the current, longitudinal Lorentz covariance excludes a constant longitudinal vector. Rotation covariance allows $\partial_{i}\delta_{\mathscr{W}}$ and $\varepsilon_{i}{}^{k}\partial_{k}\delta_{\mathscr{W}}$. The radial term is even under meridian reflection, whereas the current must reverse under charge conjugation. The simultaneous requirements exclude the radial term. Thus the current difference is precisely the term in .

Stress conservation gives $v + z = 0$, so write the transverse block as $v'K_{ij}\delta_{\mathscr{W}}$. Its trace is

$$(2u - v')\Delta\delta_{\mathscr{W}}.$$

The trace equation in requires

$$2u - v' = 3nq\,c.$$

Put $\tau = u - nq\,c$. Then $v' = 2\tau - nq\,c$, which is exactly by and . The parameters are unique because the listed contact generators are linearly independent.

Conversely, the contact tensors in – satisfy all the symmetries, scaling conditions, and Ward equations. The improvement identity follows by subtracting the same formula at $q = 0$. At a self-conjugate sector, oddness forces $w = 0$.

Remark (What makes the dimension finite). Without longitudinal translation invariance, the coefficients of the normal delta derivatives could themselves be distributions on $\mathscr{W}$, giving an infinite-dimensional ambiguity. Without , the scaling-degree bound also permits lower-order contacts, including a longitudinal stress proportional to $L^{-2}\delta_{\mathscr{W}}$. Without the reflection and charge-conjugation requirements, additional oriented tensor structures can occur. The dimension in Theorem is not a universal dimension for every apex prescription.

Remark (No physical normalization is selected). The Friedrichs condition fixes the spatial wave operator on the regular host. It does not fix c, τ, w. A specified defect action, core model, variational prescription, or independent apex normalization could impose additional conditions. None is introduced here; Section proves that within this class none can be canonical.

Proposition (Reduced hypotheses; correction E3). *Drop transverse rotation covariance, meridian reflection, charge conjugation and almost homogeneity from Assumption . Keep:*

• *the scaling-degree bounds* $2, 4, 3$*;*

• *longitudinal translation invariance and Lorentz covariance;*

• $\bar{g}$*-symmetry;*

• *the Ward identities . The supported differences are then still exactly the three-parameter family* $(c\delta_{\mathscr{W}},\ \tau\mathscr{H} + nqc\,\mathscr{I}[\delta_{\mathscr{W}}],\ w\,\varepsilon\partial\delta_{\mathscr{W}})$*. With a mass term in the trace identity, the same holds with the modification of*

Theorem (v). Without $\bar{g}$-symmetry the dimension rises to 8. *For the Killing–homothety identities of Definition , the dimensions are those of Table .*

Proof. A difference is $\sum_{|\gamma|\le k} c_\gamma \partial^\gamma \delta_{\mathscr{W}}$ with constant coefficients (translation invariance), $k = 0, 2, 1$. In the Fourier picture $\partial_i \delta_{\mathscr{W}} \leftrightarrow d_i$, multiplication by x_l acts as $-\partial/\partial d_l$. Each Ward identity then becomes a polynomial identity in (d_1, d_2), and the solution space follows by exact linear algebra (code/class_sym.py).

• Lower-order terms are excluded by conservation and the trace. For example, $\delta_{\mathscr{W}} \delta^i_j$ has divergence $\partial_j \delta_{\mathscr{W}} \neq 0$. The term $L^{-2} \delta_{\mathscr{W}} \delta^A_B$ has trace $2L^{-2}\delta_{\mathscr{W}}$, which is not of the form $3nq\, c\, \Delta \delta_{\mathscr{W}}$.

• The radial current contact $\partial_i \delta_{\mathscr{W}}$ has divergence $\Delta \delta_{\mathscr{W}} \neq 0$.

• The odd symmetric tensor $\partial_i (\varepsilon\partial)_j \delta_{\mathscr{W}} + \partial_j (\varepsilon\partial)_i \delta_{\mathscr{W}}$ has divergence $\Delta (\varepsilon\partial)_j \delta_{\mathscr{W}} \neq 0$.

Dilation residues and compatibility of presentations

The unavoidable scalar residue

Proposition (Euler identity and scale dependence). *The generating finite part obeys*

$$(E + 2) U_L = \sigma_\alpha \delta_{\mathscr{W}}, \qquad L \frac{\partial U_L}{\partial L} = -\sigma_\alpha \delta_{\mathscr{W}},$$

and hence

$$U_{L'} - U_L = -\sigma_\alpha \log (L'/L) \delta_{\mathscr{W}}.$$

For the transverse dilation $x \mapsto \lambda x$,

$$U_L(\lambda x) = \lambda^{-2} \big(U_L(x) + \sigma_\alpha \log \lambda\, \delta_{\mathscr{W}} \big)$$

as a pullback identity of generalized functions. No extension of r^{-2} whatsoever, of any order and without translation invariance, is strictly homogeneous (sharpened in version 2, E5; proof in Theorem).

Proof. For a test *f*,

$$\langle (E+2)U_L, f\rangle = -\text{langleU_}L, Ef\rangle.$$

The subtraction term vanishes because $Ef(y,0) = 0$. Radial integration gives

$$\lim_{\varepsilon \text{downarrow } 0} \alpha \int_{r>\varepsilon} r^{-2} Ef \, \mathrm{d}^2x = -\sigma_\alpha f(y,0)$$

after the longitudinal integration. This proves the first identity. Differentiating the defining subtraction proves the second, and integration in L gives . Integrating the Euler equation along dilations gives the pullback formula. Since $(E+2)\delta_{\mathscr{W}} = 0$, adding a scalar contact cannot remove the residue.

Corollary (Simultaneous residues). *For the canonical prescription,*

$$(E+2)\widehat{F}_L = s\sigma_\alpha \delta_{\mathscr{W}},$$

$$(E+4)(\widehat{T}_L)^A{}_B = \frac{a\sigma_\alpha}{4}\delta^A_B \Delta\delta_{\mathscr{W}},$$

$$(E+4)(\widehat{T}_L)^i{}_j = \frac{b\sigma_\alpha}{2}K_{ij}\delta_{\mathscr{W}},$$

$$(E+3)(\widehat{J}_L)^i = -\frac{j\sigma_\alpha}{2\alpha^2}\varepsilon^i{}_k \partial_k \delta_{\mathscr{W}}.$$

These residues are unchanged by the contacts in Theorem . The L-derivative of each canonical distribution is the negative of its displayed residue.

Proof. Use $[E, \partial_i] = -\partial_i$ in , then apply the defining differential operators. Each permitted contact is exactly homogeneous at the relevant degree and is therefore killed by the corresponding Euler operator.

The residues are extension scaling defects. They must not be identified without further analysis with a curvature-dependent quantum trace anomaly. For example, at $\xi = 1/6$ the mixed trace of the constructed stress is exactly zero, even when its normal-dilation residue is nonzero. Changing L changes a contact convention and does not change any regular-part value.

The angular current is a smooth contraction, not singular lowering

The smooth contraction defining an angular scalar extension is

$$J_{\varphi}^{\mathrm{ext}} := \alpha^2 R_i J^i, \qquad R_i = \delta_{ij} R^j.$$

On M, this is the actual covariant angular component, because $g_{ij}R^i = \alpha^2 R_j$. Equation uses a smooth polynomial multiplier on the completion; it does not multiply by the singular metric there.

Proposition (A computed contraction defect). *For the derivative-compatible current,*

$$\alpha^2 R_i (\hat{J}_L)^i = jU_L - \frac{j\sigma_\alpha}{2}\delta_{\mathscr{W}}.$$

For the general current ,

$$J_{\varphi}^{\mathrm{ext}} = jU_L - \left(\frac{j\sigma_\alpha}{2} + 2\alpha^2 w\right)\delta_{\mathscr{W}}.$$

In contrast, the direct componentwise current obeys

$$\alpha^2 R_i (J_L^{\mathrm{c}})^i = jU_L.$$

Thus a contact can preserve this latter contraction identity, but not simultaneously the unmodified derivative identity when $j \neq 0$.

Proof. Since $R_i \varepsilon^i{}_k = x_k$,

$$\alpha^2 R_i (\hat{J}_L)^i = -\frac{j}{2} E U_L.$$

Equation gives . Also $x^i \partial_i \delta_{\mathscr{W}} = -2\delta_{\mathscr{W}}$, proving . Equation then gives the direct finite-part contraction. Choosing $w = -j\sigma_\alpha/(4\alpha^2)$ removes the contraction defect, but changes by that nonzero contact.

This is an explicit presentation-compatibility calculation. It does not rely on checking finitely many null relations and then assuming coverage of all relations. The distinction between a distributional extension and a descending realization of several presentations is also emphasized in the supplied XV-B material [15].

The potential extension theorem (new)

Version 1 generated all three observables from the single distribution U_L. That works because the Friedrichs data

are exactly homogeneous and share one radial profile. For scale-dependent, massive, thermal or non-Friedrichs states this fails. The construction below replaces U_L by three scalar potentials, which are extended separately.

Assumption (Rotation-invariant diagonal data). On M, let $F(r)$, $T = \mathrm{diag}\,(A, B_r, B_\varphi, A)(r)$ in (t, r, φ, z), and $J_\varphi(r)$ be smooth, with $J^r = J^A = 0$. Assume, for some $n \in \{1, 2\}$ and $m \geq 0$:

• conservation: $\partial_r(rB_r) = B_\varphi$, and the current is automatically conserved;

• the trace identity: $2A + B_r + B_\varphi = n(3q\Delta F - m^2 F)$. Assume further the near-apex hypothesis (H_δ): there are constants s, b, j and $\delta \in (0, 2]$ with

$$F - \frac{s}{r^2}, \quad r^2\Big(B_r - \frac{b}{r^4}\Big), \quad J_\varphi - \frac{j}{r^2} \;=\; O(r^{-2+\delta}(1 + |\log r|)) \qquad (r \to 0).$$

Put $a = b + 6nqs$.

Define the potentials

$$\Psi(r) = -\int_{r_0}^{r} \rho\, B_r(\rho)\, \mathrm{d}\rho, \qquad Z(r) = \int_{r_0}^{r} \frac{J_\varphi(\rho)}{\alpha^2 \rho}\, \mathrm{d}\rho.$$

A change of r_0 adds constants, which are annihilated below. On M,

$$T^i{}_j = K_{ij}\Psi, \qquad T^A{}_B = \delta^A_B[\tfrac{1}{2}\Delta(\Psi + 3qnF) - \tfrac{1}{2}nm^2F], \qquad J^i = \varepsilon^i{}_k\partial_k Z.$$

Indeed, $K_{rr}\Psi = -\Psi'/r = B_r$ and $K_{\varphi\varphi}\Psi = -\Psi'' = (rB_r)'$. The trace identity determines A. Finally $\varepsilon\partial Z = Z' R/r$ and $J_\varphi = \alpha^2 r Z'$.

By (H_δ), $\Psi = \frac{b}{2r^2} + O(r^{-2+\delta}(1 + |\log r|))$ and $Z = -\frac{j}{2\alpha^2 r^2} + O(\cdots)$. Each of F, Ψ, Z is therefore a multiple of r^{-2} plus a function in $L^1_{\mathrm{loc}}(\overline{M}, \mathrm{d}\Omega)$.

Definition (Potential extension). For $G \in \{F, \Psi, Z\}$ with leading coefficient $g \in \{s, b/2, -j/2\alpha^2\}$, put $\widehat{G}_L = g\,U_L + (G - g\,r^{-2})$, the second term as an L^1_{loc} distribution. Define $\widehat{F}_L$,

$$(\widehat{T}_L)^i{}_j = K_{ij}\widehat{\Psi}_L, \qquad (\widehat{T}_L)^A{}_B = \delta^A_B[\tfrac{1}{2}\Delta(\widehat{\Psi}_L + 3qn\widehat{F}_L) - \tfrac{1}{2}nm^2\widehat{F}_L], \qquad \widehat{J}^i_L = \varepsilon^i{}_k\partial_k\widehat{Z}_L.$$

Theorem (Potential extension). *Under Assumption :*

1. *$(\widehat{F}_L, \widehat{T}_L, \widehat{J}_L)$ are real extensions of (F, T, J). Their transverse scaling degrees are at most $2, 4, 3$, and they are rotation invariant, stationary and boost invariant.*

2. *The Ward identities hold exactly in both forms ($\mathfrak{D}$ and W1–W3):*

$$\partial_i (\widehat{T}_L)^i{}_j = 0, \qquad \partial_i \widehat{J}_L^{\,i} = 0, \qquad (\widehat{T}_L)^\mu{}_\mu = n(3q\Delta\widehat{F}_L - m^2\widehat{F}_L).$$

The improvement identity holds whenever the data obey it on M. The $\bar{g}$-lowered stress is symmetric.

3. *If $m = 0$ and the data are exactly homogeneous ($F = sr^{-2}$, $T = r^{-4}\mathrm{diag}(a, b, -3b, a)$, $J_\varphi = jr^{-2}$), the construction reproduces Theorem .*

4. *Joint residues. Suppose the data depend on lengths $\ell_1, \dots, \ell_k$ (for example m^{-1}, M_λ^{-1}, T^{-1}). Suppose F, T, J are jointly homogeneous of degrees $-2, -4, -3$ under $(x, \ell) \mapsto (\lambda x, \lambda\ell)$, and put $\mathbb{E} = E + \sum_a \ell_a \partial_{\ell_a}$. Then, at fixed L,*

$$(\mathbb{E} + 2)\widehat{F}_L = s\sigma_\alpha \delta_{\mathscr{W}}, \qquad (\mathbb{E} + 4)(\widehat{T}_L)^i{}_j = \tfrac{b}{2}\sigma_\alpha K_{ij}\delta_{\mathscr{W}},$$

$$(\mathbb{E} + 3)\widehat{J}_L^{\,i} = -\tfrac{j}{2\alpha^2}\sigma_\alpha \varepsilon^i{}_k \partial_k \delta_{\mathscr{W}},$$

$$(\mathbb{E} + 4)(\widehat{T}_L)^A{}_B = \delta^A_B[\tfrac{a}{4}\sigma_\alpha \Delta\delta_{\mathscr{W}} - \tfrac{1}{2}nm^2 s\,\sigma_\alpha \delta_{\mathscr{W}}].$$

The last term is a new mass residue.

5. *Contacts. In the class of Proposition with the trace identity of (ii), every extension is uniquely*

$$(\widehat{F}_L + c\delta_{\mathscr{W}},\ \widehat{T}_L + \tau\mathscr{H} + nqc\,\mathscr{I}[\delta_{\mathscr{W}}] - \tfrac{1}{2}nm^2 c\,\delta^A_B\delta_{\mathscr{W}},\ ?\widehat{J}_L + w\,\varepsilon\partial\delta_{\mathscr{W}}),$$

with three real parameters.

6. *Exactly homogeneous members. If ℓ is one of the lengths and $L = \kappa\ell$ with fixed $\kappa > 0$, then $(\mathbb{E} + d)$ annihilates all three extensions. Changing κ moves along the generator line of Theorem .*

Proof. (i) On *M* the definitions reduce to . The scaling-degree bounds follow from those of U_L and from local

integrability of the remainders.

(ii) $\partial_i K_{ij} = 0$ and $\partial_i \varepsilon^i{}_k \partial_k = 0$ hold on all distributions. The trace is $\Delta(\widehat{\Psi} + 3qn\widehat{F}) - nm^2\widehat{F} + K_{ii}\widehat{\Psi}$, and $K_{ii} = -\Delta$. Symmetry is evident. By Lemma , W1–W3 follow.

(iii) $\widehat{\Psi}_L = \frac{b}{2}U_L$ and $\widehat{Z}_L = -\frac{j}{2\alpha^2}U_L$. Also $\frac{1}{2}\Delta(\frac{b}{2} + 3qns)U_L = \frac{a}{4}\Delta U_L$.

(iv) A remainder $R = G - gr^{-2}$ is L^1_{loc} and jointly homogeneous of degree −2. Pullback by dilations commutes with the pairing for L^1 functions, so $(\mathbb{E} + 2)R = 0$ as a distribution. With this gives $(\mathbb{E} + 2)\widehat{G}_L = g\sigma_\alpha \delta_{\mathscr{W}}$. Then use the intertwinings $(E + 4)\Delta = \Delta(E + 2)$, $(E + 4)K_{ij} = K_{ij}(E + 2)$, $(E + 3)\partial_k = \partial_k(E + 2)$ and $\mathbb{E}\, m^2 = -2m^2$. For the mass term, $(\mathbb{E} + 4)(m^2\widehat{F}) = m^2(\mathbb{E} + 2)\widehat{F}$.

(v) The difference of two extensions is supported and of the form treated in Proposition . With a mass term, the trace forces the order-0 longitudinal contact $-\frac{1}{2}nm^2 c\, \delta^A_B \delta_{\mathscr{W}}$ and nothing else (class_sym, massive row: $u_0 = -m^2 n(\dots)$).

(vi) $L\partial_L U_L = -\sigma_\alpha \delta_{\mathscr{W}}$, and at $L = \kappa\ell$, $\ell\partial_\ell$ also acts on L. This cancels the residues in (iv).

Remark. Theorem uses only the leading coefficients (s, b, j) near the apex, so the residues are *local apex data*. They do not depend on the large-r behaviour or on the scales, except through the explicit mass term. The proof does not need $T^t{}_t = T^z{}_z$ beyond its use in . For thermal states, where boost invariance fails, Section uses a different and simpler argument.

Non-Friedrichs nonnegative extensions (ledger item (a))

The regular data are the closed forms of I-B-VP-SAE [32] (Theorems thm:shift, thm:line, thm:Elam, Proposition prop:u2). There the field is complex ($n = 2$) and the corrections are relative to the Friedrichs state of the same sector. We use the current normalisation of with $e = 1$. The convention of I-B-VP-SAE agrees with it: the shift identities below reproduce I-B-VP-SAE's correction $\delta j_\varphi = -q|q|/(2\pi^2\alpha^2 r^2)$ exactly.

E0 and (E0,E0): version 1 at shifted flux

Theorem (Apex finite parts of the E_0 states). *Write $P_\beta, C_\beta, j_\beta$ for the polynomials in and , and let $(s,h,a,b,j)[\beta']$ be the coefficients obtained by substituting $\beta \mapsto \beta'$. The regular data of $\mathrm{E}_0^{(0)}$ ($\beta < \alpha$), $\mathrm{E}_0^{(-1)}$ ($1-\beta < \alpha$) and $(\mathrm{E}_0^{(0)}, \mathrm{E}_0^{(-1)})$ are exactly homogeneous of the version 1 form. Their coefficients are*

$$\mathbf{c}[\beta+1], \qquad \mathbf{c}[\beta-1], \qquad \mathbf{c}[\beta+1]+\mathbf{c}[\beta-1]-\mathbf{c}[\beta], \qquad \mathbf{c}=(s,h,a,b,j).$$

Hence Theorems , , and the residues hold verbatim with these coefficients. Explicitly,

$$s_{\mathrm{E}_0^{(0)}} = \frac{1+6\beta+6\beta^2-\alpha^2}{48\pi^2\alpha^2}, \qquad j_{\mathrm{E}_0^{(0)}} = -\frac{\beta(1+\beta)(1+2\beta)}{12\pi^2\alpha^2},$$

$$s_{\mathrm{E}_0^{(-1)}} = \frac{13-18\beta+6\beta^2-\alpha^2}{48\pi^2\alpha^2}, \qquad j_{\mathrm{E}_0^{(-1)}} = \frac{(1-\beta)(2-\beta)(3-2\beta)}{12\pi^2\alpha^2},$$

$$s_{(\mathrm{E}_0,\mathrm{E}_0)} = \frac{13-6\beta+6\beta^2-\alpha^2}{48\pi^2\alpha^2}, \qquad j_{(\mathrm{E}_0,\mathrm{E}_0)} = -\frac{(2\beta-1)(\beta^2-\beta+6)}{12\pi^2\alpha^2}.$$

In each case $s > 0$. Consequently the field-square residue $s\sigma_\alpha\delta_{\mathscr{W}}$ never vanishes for an E_0-type state, and no extension of the field square is a signed measure. For the Friedrichs state, by contrast, the field-square residue vanishes exactly on the locus $\alpha^2 = 1-6\beta(1-\beta)$. There $F^\circ = 0$ and the field-square extension is just $c\delta_{\mathscr{W}}$.

Proof. I-B-VP-SAE Theorem thm:shift gives the shift identities. Sympy confirms (log ledger_a, A1) that the shifted polynomials minus the Friedrichs ones equal I-B-VP-SAE's explicit E_0 corrections for all four of s,a,b,j, with exact residual 0. It also confirms that $a-b-6nqs = 0$ for the shifted coefficients. Positivity holds because $1+6\beta+6\beta^2-\alpha^2 > 0$, because $6\beta^2-18\beta+13 \geq 1$ on $[0,1]$, and because $6\beta^2-6\beta+13 \geq \frac{23}{2}$.

Scale-dependent E-lambda

Theorem (Apex extension of E_λ). *Let E_λ ($0 < \lambda < \infty$) act in one mode with $\nu = \nu_n \in (0,1)$, and let M_λ be its scale. Then Assumption holds with $m = 0$ and every $\delta < \min(2\nu, 2-2\nu)$. The near-apex coefficients (s,b,j) are those of the corresponding E_0 state. Hence:*

1. *the potential extension exists and satisfies all Ward identities;*

2. *the joint residues , with* $\ell = M_\lambda^{-1}$*, are the scale-independent* E_0 *residues;*

3. *the contact space is three-dimensional;*

4. $L = \kappa/M_\lambda$ *gives a one-parameter family of jointly exactly homogeneous extensions.*

Proof. In I-B-VP-SAE Theorem thm:Elam, each correction is r^{-2} or r^{-4} times $\int_0^\infty x\, w_\varepsilon(x)\, t(x)\, \mathrm{d}x$, with weight $w_\varepsilon = x^{2v}/(x^{2v} + \varepsilon^{2v})$ and $\varepsilon = M_\lambda r$. At $\varepsilon = 0$ the weight is 1, which is the E_0 value. For $0 \le \delta' \le 2v$,

$$0 \le 1 - w_\varepsilon(x) = \frac{\varepsilon^{2v}}{x^{2v} + \varepsilon^{2v}} \le \left(\frac{\varepsilon}{x}\right)^{\delta'}.$$

The integrands $t(x)$ are $O(x^{-2v})$ at 0 and decay exponentially. So $\int x^{1-\delta'}|t|\,\mathrm{d}x < \infty$ whenever $\delta' < 2 - 2v$, and the remainder is $O(\varepsilon^{\delta'})$. The current is $-2eq$ times the field-square correction. Theorem then applies.

Numerically, at $\alpha = 0.8$ and $\xi = 0$ (logs/elam.json), the measured remainder exponents are as follows.

• $v = 0.3$: $1 - \Phi_v$ gives $0.576,\ 0.593$, compared with $2v = 0.6$; $r^4 \delta T^r{}_r$ −(E_0) gives $0.591,\ 0.598$.

• $v = 0.75$: $1 - \Phi_v$ gives $0.4993,\ 0.49993$, compared with $2 - 2v = 0.5$; for $T^r{}_r$ the ε^{2-2v} coefficient vanishes and the exponent is $1.476,\ 1.492$, close to $2v = 1.5$. All exponents are positive, as the proof requires.

Non-diagonal U(2) extensions

For $H \ge 0$ with $H_{0,-1} \ne 0$ (I-B-VP-SAE Proposition prop:u2), the data depend on φ through $\ell = \pm 1$ harmonics. As $r \to 0$ they tend to the $(\mathrm{E}_0, \mathrm{E}_0)$ values.

Proposition (Conditional extension). *Suppose* $C(p) - \mathrm{diag}\,(c_0(v_0), c_0(v_{-1})) = O(p^{-\delta_0})$ *as* $p \to \infty$*, entrywise, for some* $\delta_0 > 0$*. Then:*

• *the rotation-averaged (*$\ell = 0$*) part satisfies (H_δ) with the* $(\mathrm{E}_0, \mathrm{E}_0)$ *coefficients, and Theorem applies;*

• *the* $\ell = \pm 1$ *part has transverse scaling degree* < 4 *for the stress and* < 2 *for the field square, with vanishing zeroth angular moment;*

• *hence the extension exists, Ward-compatible in the W1–W3 sense, and its residues are those of* $(\mathrm{E}_0, \mathrm{E}_0)$.

Proof. The bound makes the $\ell = 0$ remainder $O(r^{-2+\delta})$ exactly as in Theorem . An $\ell = \pm 1$ function of order $r^{-4+\delta}$ has a convergent angular-first pairing, because $\int A\, d\varphi = 0$ and the first-order Taylor term is integrable. Its Ward defects are supported and of scaling degree < 5. Imposing W1–W3 on the pairing reduces them to the finite contact algebra of Table .

The status is **open with a precise obstruction**: the large-p bound on the off-diagonal Krein entry is not proved here. I-B-VP-SAE supports it only numerically (relative error 2×10^{-4} at $r = 10^{-6}$).

The beta=1/2 line family

Theorem (Line family: structure, failure of $\mathfrak{D}$, and extension). *Let $\beta = \frac{1}{2}$ and $\alpha > \frac{1}{2}$, and take the I-B-VP-SAE state E_0 on $v = (\cos\theta, \sin\theta e^{i\chi})$. Write $u = \sin 2\theta \cos(\varphi - \chi)$, $\pi = \sin 2\theta \sin(\varphi - \chi)$ and $k_\xi = 6\xi - 1$.*

1. *Isotypic splitting. The data split into an $\ell = 0$ part and an odd ($\ell = 1$) part. The $\ell = 0$ part is independent of (θ, χ) for F and T, and equals the single-E_0 correction at $\beta = \frac{1}{2}$. The current coefficient is $j = -\cos 2\theta/(8\pi^2\alpha^2)$. The odd part is an exact improvement term:*

$$T_{\mathrm{odd}} = nq\,(g\mathrm{Box}_g - \nabla^g\nabla^g)F_{\mathrm{odd}}, \qquad F_{\mathrm{odd}} = \frac{u}{8\pi^2\alpha^2 r^2}, \quad n = 2,$$

with $\mathrm{Box}_g(r^{-2}\cos\varphi) = (4 - \alpha^{-2})r^{-4}\cos\varphi$. *It vanishes at $\xi = \frac{1}{6}$.*

2. *Failure of the version 1 representative. The $\bar{g}$-lowered stress is not symmetric:*

$$T^r{}_\varphi - r^2 T^\varphi{}_r = -(1 - \alpha^2)\alpha^{-2} T^r{}_\varphi = \frac{(1 - \alpha^2)k_\xi \pi}{8\pi^2\alpha^4 r^3}.$$

The flat divergence is $\partial_i T^i{}_j = (1 - \alpha^2)\omega_j^\perp T^\varphi{}_r \neq 0$ on M, although $\nabla^g T = 0$. So Definition does not apply (correction E7).

3. *Killing–homothety identities W1–W3 and the trace identity $T^\mu{}_\mu = 6q\,\mathrm{Box}_g F$ hold on M.*

4. *Extension.*

• *The $\ell = 0$ part is extended by Theorem with the E_0 coefficients at $\beta = \frac{1}{2}$ and $j = -\cos 2\theta/(8\pi^2\alpha^2)$.*

• *The odd part has a unique parity-odd extension; it is homogeneous of degrees* −2 *(field square) and* −4 *(stress).*

• *The sum satisfies W1–W3 and the trace identity distributionally, where* $\mathrm{Box}_g F_{\mathrm{odd}}$ *is extended as its unique odd extension.*

• *The residues are those of the* $\ell = 0$ *part, which coincide with those of* E_0 *at* $\beta = \frac{1}{2}$.

5. *Contacts. In the W1–W3 class with homogeneity and without rotation covariance the contact space has dimension* 8*. If the difference is required to be rotation covariant and meridian-reflection even, it has dimension* 3 *(Table).*

Proof. (i)–(iii) are symbolic identities (logs/ledger_a.json, A2), all with exact residual 0:

• the improvement identity for each of the five odd components;

• $\nabla^g T = 0$, W2, W3 and the trace identity;

• the $\ell = 0$ averages against I-B-VP-SAE thm:shift;

• parity of the odd Cartesian components. The $\bar{g}$-divergence comes out as $(1 - \alpha^2)\omega_j^{\perp} T^{\varphi}{}_r$ exactly.

(iv) Under $x \mapsto -x$ the odd part is odd. For degree −2, its zeroth moment $\int u \, \mathrm{d}\varphi$ vanishes. For degree −4, its second moments $\int A \, \omega_i \omega_j \, \mathrm{d}\varphi$ vanish because A has only odd harmonics. So the finite part has no logarithm and is homogeneous. Two odd extensions differ by an odd supported distribution of degree −2 or −4. Such distributions are combinations of $\delta_{\mathscr{W}}$ or $\partial\partial\delta_{\mathscr{W}}$, which are even, so the difference is 0.

The defects of W2, W3 and the trace identity for the odd part are odd, supported and homogeneous of degree −4. They vanish for the same reason. Note that $T \cdot R$ and $T \cdot X$ are even of degree −3, so their divergences are odd of degree −4. The $\ell = 0$ part is diagonal and rotation invariant, so Lemma applies. (v) is the exact computation behind Table .

Remark.

• At $\beta = \frac{1}{2}$ the state with $\cos 2\theta \neq 0$ is not charge-conjugation invariant. So w is not forced to vanish, although the

sector is self-conjugate.

• Item (i) gives a structural reading of the (θ,χ)-independence at $\xi = \frac{1}{6}$ found in I-B-VP-SAE: the odd part is a pure improvement term.

The thermal Friedrichs state (ledger item (b))

Let ω_T be the KMS state at temperature T of the Friedrichs dynamics in a holonomy sector β. Its Euclidean Green function is the image sum in $\tau \mapsto \tau + k/T$ of the vacuum function [24, 23, 21]. At $\Delta\varphi = 0$ and $r = r'$, the vacuum Euclidean Friedrichs function per complex component is

$$G(w;r) = \frac{\mathcal{I}(\eta/\alpha)}{8\pi^2\alpha r^2 \sinh\eta}, \qquad \cosh\eta = 1 + \frac{w}{2r^2}, \qquad \mathcal{I}(x) = \frac{\cosh((\frac{1}{2}-\beta)x)}{\sinh(x/2)},$$

with $w = \Delta\tau^2 + \Delta z^2$. This is the closed form of the I-B-VP-SAE shift argument. At $\alpha = 1$, $\beta = 0$ it is $1/(4\pi^2 w)$.

Theorem (Thermal one-point functions and their apex extension). *Put $g_k' = \partial_w G|_{w = k^2/T^2}$ and similarly g_k''. Write $\delta X = X_{\omega_T} - X_{\mathrm{vac}}$. Then on M:*

$$\delta F = 2\sum_{k\geq 1} G(k^2/T^2;r), \qquad \delta C^{\tau}{}_{\tau} = 2\sum_{k\geq 1}(2g_k' + 4w_k g_k''), \qquad \delta C^{z}{}_{z} = 2\sum_{k\geq 1} 2g_k',$$

$$\delta T^{t}{}_{t} = n[-\delta C^{\tau}{}_{\tau} + (\xi - \tfrac{1}{4})\Delta\delta F], \qquad \delta T^{z}{}_{z} = n[-\delta C^{z}{}_{z} + (\xi - \tfrac{1}{4})\Delta\delta F],$$

$$\delta T^{r}{}_{r} = \frac{1}{r^2}\int_0^r \rho(3qn\Delta\delta F - \delta T^{t}{}_{t} - \delta T^{z}{}_{z})\mathrm{d}\rho, \qquad \delta T^{\varphi}{}_{\varphi} = \partial_r(r\,\delta T^{r}{}_{r}),$$

where $C^{\mu}{}_{\mu} = \langle\varphi\,\partial_\mu^2\varphi\rangle$ (no sum). Moreover:

1. *Apex behaviour. For $\beta = 0$, $\delta F \to T^2/(12\alpha)$ as $r \to 0$. For $0 < \beta < 1$, $\delta F = O(r^{2\nu_*})$ with $\nu_* = \min(\beta, 1-\beta)/\alpha$. All stress and current corrections are $O(r^{-2+2\nu_*})$ in Cartesian components, or bounded when $\beta = 0$. In particular they are in $L^1_{\mathrm{loc}}(\overline{M})$.*

2. *Extension. $\widehat{X}_T = \widehat{X}_{\mathrm{vac},L} + \delta X$, with δX taken as L^1 functions, satisfies the version 1 Ward identities and the trace and improvement identities distributionally. Its joint residues ($\ell = 1/T$) are the vacuum residues .*

3. *Contacts. Boost invariance is lost. Impose stationarity, z-translation, z-reflection, CT, rotation covariance,* $\bar{g}$ *-symmetry, the scaling bounds and the Ward identities. The supported differences are then*

$$c\delta_{\mathscr{W}}; \qquad \tau_t, \tau_z \text{ in } T^t{}_t, T^z{}_z \ (\times \Delta\delta_{\mathscr{W}}); \qquad \lambda \operatorname{diag}(1,-1)\delta_{\mathscr{W}} \text{ in } T^A{}_B;$$

$$\nu' K_{ij}\delta_{\mathscr{W}}; \qquad w\,\varepsilon\partial\delta_{\mathscr{W}},$$

with the single trace relation $\tau_t + \tau_z - \nu' = 3nqc$*. The dimension is* 5 *(generic charged) or* 4 *(neutral). The traceless anisotropic contact* $\lambda \operatorname{diag}(1,-1)\delta_{\mathscr{W}}$ *(dimension* T^2*) is new. Boost invariance excludes it in the vacuum.*

Proof. The formulas follow from the image sum and the point-split stress $T_{\mu\nu} = (1-2\xi)\partial_\mu\varphi\partial_\nu\varphi + (2\xi - \frac{1}{2})g_{\mu\nu}(\partial\varphi)^2 - 2\xi\varphi\nabla_\mu\nabla_\nu\varphi$. They use $\langle\partial\varphi\cdot\partial\varphi\rangle = \frac{1}{2}\Delta F$ (massless) and the conservation and trace on M. The integration constant in $\delta T^r{}_r$ is fixed by regularity at $r = 0$.

(i) Expand in modes, with the ν_n of the sector. Each image term is a bisolution at Euclidean separation $k/T > 0$, and its mode functions are $O((rr')^{\nu_n})$ near $r = 0$. So coincidence limits are $O(r^{2\nu_*})$, and two derivatives cost r^{-2}. For $\beta = 0$ the $n = 0$ mode is smooth and gives the constant

$$2\sum_k \frac{1}{4\pi^2\alpha k^2 T^{-2}} = \frac{T^2}{12\alpha}.$$

(ii) An L^1_{loc} function f with $|\nabla f| = O(r^{-1+2\nu_*})$ has distributional derivative equal to its pointwise derivative. The boundary term on $r = \varepsilon$ is $O(\varepsilon^{2\nu_*})$. For the stress the flux is $\oint T^r{}_r\,\omega_j\,\varepsilon\,\mathrm{d}\varphi = 0$ by rotation invariance. So the Ward identities pass to the distributions. Joint homogeneity in (x, T^{-1}) makes $(\mathbb{E} + d)\delta X = 0$.

(iii) Enumerate as in Proposition . Without boosts, $T^A{}_B$ admits independent *tt* and *zz* entries. The *tz* entry is excluded by *z*-reflection. The order-0 entries are constrained only by the trace. Mixed $T^t{}_i$ contacts are either excluded by conservation ($\partial_i\delta_{\mathscr{W}}$) or *CT*-odd ($\varepsilon\partial\delta_{\mathscr{W}}$).

Checks against the literature (numbers from logs/thermal.json).

• *Flat limit* ($\alpha = 1$, $\beta = 0$, $T = 1$): $\delta F - T^2/12 = 0$, $T^t{}_t + \pi^2/30 = -5.6\times10^{-17}$, $T^z{}_z - \pi^2/90 = 1.4\times10^{-17}$.

• *Field square against Guimarães* [21], eq. (18), at $T = 1$. Guimarães' B is our α and his γ is our β; his $\langle\varphi^2\rangle$ is our total F, vacuum plus thermal. The relative deviations are $\le 1.0 \times 10^{-10}$ for $\beta = 0$, and $\le 6.1 \times 10^{-7}$ for $\beta = 0.3$, where the slowly converging image tail is summed with acceleration at 15 digits at (α, β, r) ranging over $\{(0.75, 0), (0.6, 0), (0.8, 0.3)\} \times \{0.05, 0.4, 1.5\}$.

• *Stress against Frolov–Pinzul–Zelnikov* [23], as quoted by Moreira [22], eqs. (7)–(9), for $\beta = 0$. FPZ use signature $(+,-,-,-)$, and in our conventions mixed components change sign. At $r = 0.4$, $T = 1$ and $(\alpha, \xi) = (0.75, 0)$, $(0.6, 0.3)$, the maximal relative deviation of ours + FPZ is 2.5×10^{-10} and 1.3×10^{-8}, respectively. For example, at $(0.75, 0)$,

$$(T^t{}_t, T^r{}_r, T^\varphi{}_\varphi, T^z{}_z)_{\text{ours}} = (-0.3774858607,\ 0.1488539722,\ 0.0211826208,\ 0.0902586767),$$

which is minus the FPZ value to the digits shown.

• *High temperature.* $F_{\text{tot}} \simeq \frac{T^2}{12} + \frac{T\mathscr{M}}{r}$, with

$$\mathscr{M} = \frac{1}{8\pi^2}\int_0^\infty \frac{\mathscr{I}(\eta/\alpha)/\alpha - \coth(\eta/2)}{\sinh(\eta/2)}\,\mathrm{d}\eta.$$

This agrees with Guimarães' $\mathscr{M}$ in eqs. (20)–(21): $\mathscr{M} = 0.01513614476820314$ (both, $\alpha = 0.7, \beta = 0$) and -0.02387963183452523 (both, $\alpha = 0.8, \beta = 0.3$), with all 16 digits equal. At $\alpha = \frac{1}{3}$ it equals the image value $\sum_{k=1,2} (8\pi \sin(\pi k/3))^{-1}$: 0.09188814923696534 for both, with all 16 digits equal.

• *Apex limits.* $12\alpha\,\delta F/T^2 \to 1$ for $\beta = 0$: $0.99987,\ 1 - 1.3 \times 10^{-8},\ 1 - 1.3 \times 10^{-12}$ at $r = 10^{-2}, 10^{-4}, 10^{-6}$. For $\beta = 0.3$, $\alpha = 0.7$: $\delta F / r^{2\beta/\alpha}$ tends to a constant, $0.08962,\ 0.089237,\ 0.089235$. Davies–Sahni [27] and Linet [24] treat thermal effects and the Euclidean thermal Green function on the cone. We compare with them only bibliographically, ^‡. Frolov–Serebriany [26] is the vacuum reference for the $\beta = 0$ coefficients.

The massive Friedrichs baseline (ledger item (c))

Theorem (Proper-time representation). *For mass $m > 0$, let ω_m be the Friedrichs ground state in sector β. Put*

$$W(z) = e^{-z}\frac{1}{\alpha}\sum_{n\in\mathbb{Z}} I_{|n+\beta|/\alpha}(z) - 1, \qquad V(z) = e^{-z}\frac{1}{\alpha}\sum_{n\in\mathbb{Z}} (n+\beta) I_{|n+\beta|/\alpha}(z), \qquad \mu = \frac{m^2 r^2}{2}.$$

Then, per field component,

$$F = \frac{1}{8\pi^2 r^2}\int_0^\infty W(z) e^{-\mu/z}\,\mathrm{d}z, \qquad C := \langle \varphi\, \partial_\tau^2 \varphi\rangle = -\frac{1}{8\pi^2 r^4}\int_0^\infty z W(z) e^{-\mu/z}\,\mathrm{d}z,$$

$$T^t{}_t = T^z{}_z = n[-C + (\xi - \tfrac{1}{4})\Delta F], \qquad T^\mu{}_\mu = n(3q\Delta F - m^2 F),$$

$$r^2 T^r{}_r = -n[2\int_r^\infty \rho C\,\mathrm{d}\rho - m^2\int_r^\infty \rho F\,\mathrm{d}\rho] + n\xi r F', \qquad T^\varphi{}_\varphi = \partial_r (r T^r{}_r).$$

The radial integrals are elementary in z: $\int_r^\infty \rho F = \frac{1}{8\pi^2}\int W \frac{1}{2} E_1(\mu/z)\,\mathrm{d}z$, *and* $\int_r^\infty \rho^{-3} e^{-a\rho^2}\mathrm{d}\rho$ *is* $\frac{1}{2}[e^{-ar^2} r^{-2} - a E_1(ar^2)]$. *The current is* $J_\varphi = -\frac{1}{4\pi^2 r^2}\int_0^\infty V(z) e^{-\mu/z}\mathrm{d}z$.

Proof. Use the Schwinger proper-time representation with the Friedrichs transverse heat kernel at coincidence. That kernel is $(4\pi s)^{-1} e^{-z}\alpha^{-1}\sum_n I_{\nu_n}(z)$, with $z = r^2/2s$ [28, 25]. Subtract the flat value $(4\pi s)^{-1}$ and change variables $s = r^2/2z$. The τ derivatives act on the longitudinal Gaussian ($\partial_\tau \partial_{\tau'} \to 1/2s$). The rest follows from the trace and conservation as in Section . The integration constant is fixed by decay at infinity.

Proposition (Massless limit, images, small-mass expansion).

1. $\int_0^\infty W = P_\beta/6$, $\int_0^\infty zW = C_\beta/180 + P_\beta/18$, *and* $-\frac{1}{4\pi^2}\int_0^\infty V = j$. *These recover and at* $m = 0$.

2. *For* $\beta = 0$ *and* $\frac{1}{2} < \alpha < 1$, $W(z) = -\frac{1}{\pi\alpha}\int_0^\infty e^{-z(1+\cosh u)}\frac{\sin(\pi/\alpha)}{\cosh(u/\alpha) - \cos(\pi/\alpha)}\mathrm{d}u$. *This is the* $\gamma = 0$ *case of the Guimarães–Linet kernel* $F_B(u, 0)$ *[21, 28]. At* $\alpha = 1/N$, $W = \sum_{k=1}^{N-1} e^{-2z\sin^2(\pi k/N)}$, *and F is the image sum* $\sum_k m K_1(m d_k)/(4\pi^2 d_k)$.

3. *As* $mr \to 0$,

$$F = \frac{s}{r^2} - \frac{m^2}{16\pi^2}[W_0 \log\frac{2}{m^2 r^2} + c_W] + o(m^2),$$

$$c_W = W_0(1 - \gamma_E) + \int_0^1 \frac{W - W_0}{z}\mathrm{d}z + \int_1^\infty \frac{W}{z}\mathrm{d}z.$$

Here $W_0 = W(0) = \alpha^{-1} - 1$ *for* $\beta = 0$ *and* $W_0 = -1$ *for* $\beta \neq 0$. *Also*

$$T^t{}_t = T^t{}_t|_{m=0} - \frac{nm^2 s}{2r^2} + o(m^2 r^{-2}), \qquad T^r{}_r = T^r{}_r|_{m=0} + \frac{K_r m^2}{r^2} + o(m^2 r^{-2}),$$

where the constant K_r *is global (it involves* $\int_r^\infty$*).*

4. *Assumption holds with* $\delta = 2$*, logarithmically. Theorem gives the apex extension, with the massless coefficients as residues together with the mass residue* $-\frac{1}{2} nm^2 s\, \sigma_\alpha \delta_B^A \delta_{\mathscr{M}}$*. The contact space is three-dimensional, with the order-*0 *longitudinal contact tied to c.*

Proof. (i) is the massless limit of Theorem compared with the AB coefficients. We prove it numerically below; analytically it is the Mellin transform at $s = 1, 2$ of the heat-kernel trace.

(ii) $\sum_n I_{N|n|}(z)/(1/N) = \sum_k e^{z\cos(2\pi k/N)}$, and $\int_0^\infty e^{-az - b/z}\mathrm{d}z = 2\sqrt{b/a}\, K_1(2\sqrt{ab})$.

(iii) Write $I(\mu) = \int W(e^{-\mu/z} - 1)\mathrm{d}z$. Then $I'(\mu) = -W_0 E_1(\mu) - A + o(1)$, and $E_1(\mu) = -\gamma_E - \log\mu + o(1)$. At $z = 0$, $W \to \alpha^{-1} I_0(0) - 1$ when $\beta = 0$, while for $\beta \neq 0$ every order is positive and $W \to -1$. Expand C similarly, using $\int zW(e^{-\mu/z} - 1) = -\mu\int W + o(\mu)$.

(iv) follows from (iii) and Theorem .

Checks (logs/massive.json).

• Bessel sum against the integral representation at $\alpha = 0.7$: differences $\le 3.4 \times 10^{-25}$ at $z = 0.01, 0.3, 1, 3, 8$; $W(0) = 0.428571\ldots = \alpha^{-1} - 1$ ($\beta = 0$) and -0.99999 ($\beta = 0.3$, $z = 10^{-12}$).

• Images at $\alpha = 1/3$: $\le 1.9 \times 10^{-25}$.

• Massless limits: at $\alpha = 0.7$, $\beta = 0$: $\int W = 0.1734693878 = P/6$ and $\int zW = 0.0754060808 = C/180 + P/18$. At $\beta = 0.3$: -0.2551020137 against -0.2551020408, -0.09806330696 (equal to all digits), and j from $V = -1.447445687 \times 10^{-3}$ against $-1.447445481 \times 10^{-3}$. The $\beta \neq 0$ deviations of $\sim 10^{-7}$ come from the z^{ν_*} endpoint behaviour on our fixed grid.

• Direct image point-splitting at $\alpha = \frac{1}{2}$ against the representation, for $F, T^t_t, T^r_r, T^\varphi_\varphi$ at $r = 0.3, 1.1$, $m = 1$, $\xi = 0.3$: relative deviations $\le 2.2 \times 10^{-13}$.

• Conservation at $\alpha = 0.7$, $\beta = 0.3$, $r = 0.5$: relative residual 2.5×10^{-19}.

• Small mass: at $\alpha = 0.7$, $\beta = 0$, $(F - \text{prediction})/m^2 = -3.6 \times 10^{-5}, -2.1 \times 10^{-6}$ for $mr = 0.05, 0.01$ ($c_W = -0.48247$); $\delta T^t_t r^2/m^2 = -1.09316 \times 10^{-3}, -1.09822 \times 10^{-3}$, tending to $-s/2 = -1.09851 \times 10^{-3}$; $K_r \approx 5.3 \times 10^{-4}$. At $\beta = 0.3$ ($c_W = 2.3159$) the residual decreases like $(mr)^{0.86} \approx (mr)^{2\nu_*}$, which is slower because W is not analytic at $z = 0$; $\delta T^t_t r^2/m^2 = 1.6364, 1.6244 \times 10^{-3}$, tending to $-s/2 = 1.6155 \times 10^{-3}$, and $K_r \approx -2.1 \times 10^{-3}$.

Canonical apex renormalization: selection and no-go (ledger item (d))

Fix (α, β, ξ), a massless exactly homogeneous state, and the class $\mathscr{A}$ of Theorem . By that theorem, $\mathscr{A}$ is an affine space over $\mathbb{R}^3 \ni (c, \tau, w)$. Transverse dilations $\mathscr{D}_\lambda : x \mapsto \lambda x$ act on extensions by pullback with the weights $\lambda^2, \lambda^4, \lambda^3$, and they preserve $\mathscr{A}$.

Theorem (Dilations act by translation; no covariant extension). *Under* $\mathscr{D}_\lambda$,

$$(c, \tau, w) \text{longmapsto} (c, \tau, w) + \sigma_\alpha \log \lambda \, (s, \frac{h}{4}, -\frac{j}{2\alpha^2}).$$

Consequently, if $(s, h, j) \neq 0$, *no extension in* $\mathscr{A}$ *is dilation invariant. Equivalently, there is no rule assigning to the data an extension in* $\mathscr{A}$ *that commutes with dilations. Every such rule needs an external length L or core scale, and different lengths differ by the displayed translation. If* $s \neq 0$, *no extension of* F° *in any class is homogeneous (correction E5).*

Proof. By Proposition , $\mathscr{D}^*_\lambda U_L = \lambda^{-2}(U_L + \sigma_\alpha \log \lambda \, \delta_{\mathscr{H}})$, and the contacts are exactly homogeneous. So $\mathscr{D}_\lambda$ translates $(\widehat{F}_L, \widehat{T}_L, \widehat{J}_L)$ by $\sigma_\alpha \log \lambda$ times $(s\delta_{\mathscr{H}}, \frac{a}{4}\Delta\delta_{\mathscr{H}}\delta^A_B \oplus \frac{b}{2}K\delta_{\mathscr{H}}, -\frac{j}{2\alpha^2}\varepsilon\partial\delta_{\mathscr{H}})$. Write the stress contact as $\tau\mathscr{H} + nqc\,\mathscr{I}[\delta_{\mathscr{H}}]$ with $c = s$. Then $\tau + nqs = \frac{a}{4}$, and $2\tau - nqs = \frac{b}{2}$ is consistent because $a - b = 6nqs$. Hence $\tau = \frac{a}{4} - k = \frac{h}{4}$. A translation by a nonzero vector has no fixed point.

For the last statement, any extension differs from sU_L by $\sum c_\gamma \otimes \partial^\gamma \delta_{\mathscr{M}}$, with arbitrary coefficient distributions c_γ on $\mathscr{M}$. Since $(E+2)\partial^\gamma \delta_{\mathscr{M}} = -|\gamma|\partial^\gamma \delta_{\mathscr{M}}$, these can never produce the order-0 term $s\sigma_\alpha \delta_{\mathscr{M}}$.

Theorem (Selection of a line: single-generator extensions). *Call an extension single-generator if there is one distribution G, translation invariant and of scaling degree ≤ 2 with $G|_M = r^{-2}$, such that $F = sG$, $T^A{}_B = \frac{a}{4}\delta^A_B \Delta G$, $T^i{}_j = \frac{b}{2}K_{ij}G$ and $J = -\frac{j}{2\alpha^2}\varepsilon\partial G$. That is, all regular differential presentations hold with a common generator. The single-generator extensions are exactly the dilation orbit $\{(\widehat{F}_L, \widehat{T}_L, \widehat{J}_L) : L > 0\}$. This is the affine line through the version 1 prescription with direction $(s, h/4, -j/2\alpha^2)$. If $(s,h,j) \neq 0$, the dilation group acts on it simply transitively.*

Proof. $G - U_L$ is supported, translation invariant and of scaling degree ≤ 2, so $G = U_L + c'\delta_{\mathscr{M}} = U_{L'}$ with $L' = Le^{-c'/\sigma_\alpha}$. Conversely, each $U_{L'}$ is a generator.

Theorem (Core models realize every point of the line). *Let $G_c : [0,\infty) \to \mathbb{R}$ be a core profile with $G_c = O(\rho^\delta)$ at 0 and $G_c - s = O(\rho^{-\delta})$ at ∞, and put $g_R = r^{-2}G_c(r/R)$. Then*

$$\lim_{R \,\text{downarrow}\, 0} [g_R + \sigma_\alpha s \log \frac{R}{L}\delta_{\mathscr{M}}] = sU_L + \sigma_\alpha c_G \delta_{\mathscr{M}}, \qquad c_G = \int_0^\infty \frac{G_c(\rho) - s\,\theta(\rho-1)}{\rho} d\rho.$$

The counterterm coefficient $-\sigma_\alpha s \log R$ is universal, while c_G depends on the model. Replacing G_c by $G_c(\cdot/\kappa)$ shifts c_G by $s\log\kappa$, so every point of the line is realized when $s \neq 0$. For the stress, a core profile $r^{-4}G_T(r/R)$ also produces a power divergence $\propto R^{-2}\delta^A_B\delta_{\mathscr{M}}$, a tension renormalization. Its subtraction lies outside the massless class of Assumption .

Proof. Pair with a test function and split at $r = L$. The inner integral is $\int_0^{L/R} G_c(\rho)\rho^{-1}d\rho = s\log(L/R) + c_G + o(1)$. For the stress, $\int r^{-4}G_T(r/R)\, r\, dr = R^{-2}\int \rho^{-3} G_T\, d\rho$.

Numbers (logs/core.json, α = 0. 7, s = 1).

• $G_1 = \rho^2/(1+\rho^2)$ gives $c_G = 0$. The limit residuals are $1.5\times10^{-3}, 2.2\times10^{-5}, 2.8\times10^{-7}$ at $R = 10^{-2}, 10^{-3}, 10^{-4}$.

• $G_2 = 1 - e^{-\rho^2}$ gives $c_G = \gamma_E/2 = 0.288607832450766$, matching quadrature to all digits shown. The residuals are

$1.5\times10^{-4}, 1.5\times10^{-6}, 1.5\times10^{-8}$.

• $G_3 = \rho^4/(1+\rho^4)$ gives $c_G = 0$, with residuals $2.3\times10^{-4}, 2.3\times10^{-6}, 2.3\times10^{-8}$.

Outcome for item (d).

• Within $\mathscr{A}$, a canonical (dilation-covariant) apex renormalization is **excluded** (Theorem).

• The presentation-compatibility requirement **selects one line**, parametrized by L (Theorem).

• The point on that line is fixed only by additional physics, such as a core model or a defect action (Theorem). For any specific physical core this constant remains **open**. The same holds for the massive, thermal and E_λ states, where an intrinsic scale ℓ exists: $L = \kappa\ell$ is natural, but κ is not fixed. The contacts (τ, w) off the line, and the thermal anisotropy λ, are likewise unconstrained by the regular data.

The field algebra at the apex (ledger item (e))

Let A be the Friedrichs spatial operator on $L^2(\Sigma)$, with $\Sigma = (0,\infty)_r \times S^1_\varphi \times \mathbb{R}_z$ and density $\alpha r\,\mathrm{d}r\,\mathrm{d}\varphi\,\mathrm{d}z$. Take the twisted operator for $\beta \neq 0$. Let π_ω be the GNS (Fock) representation of the ground state ω. For real $f \in C^\infty_c(M)$, the smeared field is $\varphi(f) = a(Kf) + a^*(Kf)$, where

$$Kf = \frac{1}{\sqrt{2}}\int \mathrm{d}t\, e^{\mathrm{i}tA^{1/2}} A^{-1/4} f_t, \qquad f_t = f(t,\cdot).$$

Assumption (Cited heat-kernel bound). The Friedrichs heat kernel obeys a Gaussian upper bound $0 \le e^{-sA_0}(x,y) \le Cs^{-3/2} e^{-d(x,y)^2/cs}$ on Σ. This holds for the Dirichlet form of a cone, which satisfies volume doubling and a Poincaré inequality [29]^†. For $\beta \neq 0$ use the diamagnetic inequality $|e^{-sA_\beta} f| \le e^{-sA_0}|f|$ [30]^†.

Theorem (No new linear apex degrees of freedom). *Under Assumption , K extends continuously to $L^1(\mathbb{R}_t; L^2_{\mathrm{comp}}(\Sigma))$. Hence:*

1. *$\varphi(f)$ is defined for every real $f \in C^\infty_c(\overline{M})$, with $\overline{M}$ the smooth completion of Assumption . If $f_\varepsilon = (1-\chi(r/\varepsilon))f$ with a cutoff χ, then $e^{\mathrm{i}\varphi(f_\varepsilon)} \to e^{\mathrm{i}\varphi(f)}$ strongly.*

2. *For every open* $\mathcal{O} \subset \overline{M}$*, the von Neumann algebras satisfy* $\pi_\omega(\mathfrak{W}(\mathcal{O}))'' = \pi_\omega(\mathfrak{W}(\mathcal{O} \cap M))''$.

3. *The same holds in the thermal and massive Friedrichs states, and in the* E_λ *states for which* $A^{-1/4}$ *is bounded on* L^2_{comp}.

Proof. Write $A^{-1/2} = \pi^{-1/2}\int_0^\infty s^{-1/2} e^{-sA}\mathrm{d}s$. The Gaussian bound gives $0 \le A^{-1/2}(x,y) \le C' d(x,y)^{-2}$. The Hardy–Littlewood–Sobolev inequality on the doubling space Σ (homogeneous dimension 3) gives

$$\|A^{-1/4} g\|^2 \le C'' \|g\|^2_{L^{3/2}} \le C_{\mathcal{K}} \|g\|^2_{L^2}$$

for g supported in a compact $\mathcal{K}$. Then $\|Kf\| \le \int \|A^{-1/4} f_t\|\, \mathrm{d}t$. The apex worldsheet has measure zero, so $f_\varepsilon \to f$ in $L^1(L^2)$ by dominated convergence. Weyl operators depend strongly continuously on the one-particle vector, and the symplectic form is continuous, which gives (i). Then (ii) follows from the bicommutant theorem. For (iii), the thermal state is quasi-free with one-particle operator $A^{-1/4}(1 + 2(e^{A^{1/2}/T} - 1)^{-1})^{1/2}$, which is dominated on compact supports in the same way. For mass m, replace A by $A + m^2 \ge A$.

Remark (What remains open).

• Theorem constructs the *linear* field algebra on $\overline{M}$, and shows that it adds nothing at the apex.

• The *Wick square* $\varphi^2(f)$ for f touching $\mathscr{W}$ is **open**. Its ω-normal-ordered version requires the distributional square ω_2^2 near $\mathscr{W} \times \mathscr{W}$. On $M \times M$ this is controlled by the Hadamard form. At the apex the Friedrichs two-point function contains diffracted singularities. We prove no wave-front or scaling-degree bound for them, so neither Hörmander's product criterion nor a scaling-degree extension is available. Joint homogeneity suggests the logarithmic case, scaling degree 4 equal to codimension 4, in which an extension would exist but would not be unique.

• Given such an operator, the c-number distributions $\widehat{F}_L + c\delta_{\mathscr{W}}$ are exactly its admissible expectation values.

• The restriction $\varphi|_{\mathscr{W}}$ is likewise not constructed. For $\beta = 0$, $\lim_{r,r' \to 0} G = 1/(4\pi^2 \alpha w)$ is α^{-1} times the flat Euclidean

two-point function restricted to a 2-plane.

Audit of version 1 and record of checks (new)

All scripts and logs named below are part of the version 2 record. Symbolic residuals are exact. Numerical residuals are quadrature differences: they are not enclosures and carry the evidence class of Section . They are *independent reruns*, not inherited reports.

Symbolic checks. Scripts: code/audit_sym.py, code/class_sym.py, code/class_extra.py, code/ledger_a.py. Each of the following gives exact residual 0.

• *S1.* The coefficients and against I-B-VP-AB for $n = 1, 2$, all four stress blocks and the current; the relation $a - b = 6k$; and Lemma (trace, improvement, polar conservation).

• *S2.* The Appendix boundary expansions, using a generic Taylor polynomial up to fifth order: all $\varepsilon^{-2}, \varepsilon^{-1}, \varepsilon^{0}$ coefficients, plus the angular moment identities.

• *S3.* Proposition , the raw defects and Corollary , in the contact algebra.

• *S4.* The Euler and residue relations and the contraction identity .

• *Contact dimensions* of Table , by exact linear solve.

• *Section* : the E_0 shift identities and the line-family identities.

Numerical (code/audit_num_fast.py, 20 digits). The test is $g = e^{-((x-0.3)^2+(y+0.2)^2)}(1 + \frac{x}{3} + \frac{xy}{5})$, at $\alpha = 0.7$ and $L = 1$.

• *N1 (Theorem).* For two smooth profiles and $\varepsilon = 0.1, 0.05, 0.025, 0.0125$, the ratio $(U^{\rho}_{L,\varepsilon} - U_L)(g)/\varepsilon^2$ is 3.3712, 3.3819, 3.3846, 3.3853 (ρ_1) and 4.2798, 4.2967, 4.3009, 4.3020 (ρ_2). This confirms the $O(\varepsilon^2)$ rate and profile independence.

• *N2 (Proposition).* The residuals of , and , of the raw divergence defect and of $S_{ii} + 2V$ are 5.3×10^{-12};

$(1.1, 0.5)\times10^{-16}$; $\le 1.7\times10^{-11}$; $\le 1.2\times10^{-11}$; 8.6×10^{-12} (the 4th-order finite parts are limited by cancellation at 20 digits).

• *N3 (Proposition , Corollary , Proposition).* The residuals of the Euler identity, the L-shift, the three residue formulas and both contractions are 3.4×10^{-21}, 8.2×10^{-17}, $\le 2.1\times10^{-20}$, 0 and 1.7×10^{-20}. In the first run of N2 we found an apparent 2×10^{-2} discrepancy. It came from a bug in our own fast quadrature (the analytic tail of the power subtraction was missing), not from version 1. After the fix, the residuals are as stated.

Version 1 compile. The unmodified version 1 source stops with pdflatex errors (undefined verb ||, verb | ket |, verb | bra | and verb | proj |), although in nonstop mode it still produces 19 pages. This is corrections E1 and E2.

General non-diagonal U(2) extensions (version 3, item 1)

Let $d_\beta = 2$, i.e. $\beta < \alpha$ and $1-\beta < \alpha$ (so $\alpha > \tfrac12$). The deficient modes are $n = 0, -1$, with $q_0 = \beta$, $q_{-1} = \beta - 1$, $\nu_0 = \beta/\alpha$, $\nu_{-1} = (1-\beta)/\alpha$, and $\nu_0 + \nu_{-1} = 1/\alpha \in (1,2)$. We use the I-B-VP-SAE Krein representation (Theorem thm:krein there): the Euclidean correction to Friedrichs is

$$\delta G_H = \frac{1}{4\pi^2\alpha}\int_0^\infty p\,\mathrm{d}p\, J_0(p|\Delta y|)\sum_{j,k} e^{\mathrm{i}q_j\varphi}K_{\nu_j}(pr)C_{jk}(p)K_{\nu_k}(pr')e^{-\mathrm{i}q_k\varphi'},$$

$$C(p) = (2E(p)HE(p) + S)^{-1},$$

with $E = \mathrm{diag}\,(\widehat{e}_\nu p^{-\nu})$, $\widehat{e}_\nu = s_\nu 2^\nu/\Gamma(1-\nu)$, $s_\nu = \pi/(2\sin\nu\pi)$, $S = \mathrm{diag}\,(s_\nu)$, $c_0(\nu) = 1/s_\nu$. Only $H \ge 0$ gives a state (I-B-VP-SAE thm:spec, thm:nogo). Write $R_{jk}(r) = \frac{1}{4\pi^2\alpha}\int_0^\infty p\, C_{jk}K_{\nu_j}(pr)K_{\nu_k}(pr)\,\mathrm{d}p$, so that $\delta F = R_{00} + R_{-1\,-1} + 2\mathrm{Re}\,(e^{\mathrm{i}\varphi}R_{0,-1})$, and $M(s) = \int_0^\infty x^{s-1}K_{\nu_0}K_{\nu_{-1}}\,\mathrm{d}x = \frac{2^{s-3}}{\Gamma(s)}\prod_{\pm,\pm}\Gamma(\frac{s\pm\nu_0\pm\nu_{-1}}{2})$ (continued meromorphically).

Theorem (Apex coefficients are those of the ultraviolet Lagrangian). *For every nonnegative $U(2)$ extension, $\lim_{p\to\infty} C(p) = C_{\mathrm{UV}}$ exists and the rotation-averaged ($\ell = 0$) data have the leading apex coefficients of the scale-invariant extension with Krein matrix C_{UV}:*

***1.** generic chart (H finite): $C_{\mathrm{UV}} = S^{-1}$, i.e. $(\mathrm{E}_0, \mathrm{E}_0)$, and $\|C(p) - S^{-1}\| = O(p^{-2\nu_{\min}})$;*

2. *boundary chart with Friedrichs direction w ($H = \infty$ on w, $H' = 0$ on $w^{\perp}$), $v_0 \neq v_{-1}$ and $w_0 w_{-1} \neq 0$: C_{UV} is E_0 in the mode of larger v and Friedrichs in the other, and $\|C(p) - C_{\mathrm{UV}}\| = O(p^{-|v_0 - v_{-1}|})$;*

3. *$\beta = \frac{1}{2}$: every Lagrangian is a sum of E_{λ_i} along the eigenvectors of H, and the UV Lagrangian is the dilation fixed point of I-B-VP-SAE thm:spec(iii). Consequently the hypothesis of Proposition holds with $\delta_0 = 2v_{\min}$ (generic) or $|v_0 - v_{-1}|$ (boundary), and Theorem applies to the $\ell = 0$ part with the coefficients (s, h, a, b, j) of C_{UV} (Theorem).*

Proof. (i) $C = S^{-1}(1 + 2EHES^{-1})^{-1}$ and $\|E\| = O(p^{-v_{\min}})$. (ii) Sherman–Morrison in the limit $t \to \infty$ of $H = t\,ww^{\dagger}$ gives $C = S^{-1} - S^{-1}uu^{\dagger}S^{-1}/(u^{\dagger}S^{-1}u)$ with $u = Ew$; as $p \to \infty$ the component of u with smaller v dominates, so $u/|u|$ tends to that coordinate axis and C to the projection onto the other. (iii) At $\beta = \frac{1}{2}$, $E = e_v \mathbb{1}$, so $C = \sum_i v_i v_i^{\dagger}/(2e_v^2 \lambda_i + s_v)$. The remainder estimates then give the hypothesis of Theorem exactly as in Theorem .

Theorem (Leading off-diagonal apex asymptotics).

1. *Generic chart. As $r \to 0$,*

$$R_{0,-1}(r) = \Lambda_{01}\, r^{-1/\alpha} + D'\, r^{-2+1/\alpha} + o(r^{-2+1/\alpha}),$$

$$\Lambda_{01} = \frac{\Gamma(v_0)\Gamma(v_{-1})\, 2^{1/\alpha}}{16\pi^2 \alpha}\, \kappa_{01}, \quad \kappa_{01} = \int_0^{\infty} p^{1-1/\alpha} C_{0,-1}(p)\, \mathrm{d}p,$$

$$D' = -\frac{c_0(v_0) c_0(v_{-1})\, \hat{e}_{v_0} \hat{e}_{v_{-1}}}{2\pi^2 \alpha} H_{0,-1}\, M(2 - 1/\alpha).$$

The leading coefficient κ_{01} is a global (infrared-sensitive) functional of H; the subleading D' is local and linear in $H_{0,-1}$.

2. *Boundary chart (ii) of Theorem with $v_0 < v_{-1}$: $R_{0,-1}(r) \simeq -\dfrac{c_0(v_{-1}) \hat{e}_{v_{-1}} \overline{w}_{-1}}{4\pi^2 \alpha\, \hat{e}_{v_0} \overline{w}_0} M(2 + v_0 - v_{-1})\, r^{-2+v_{-1}-v_0}$ (and symmetrically).*

3. *All one-point functions. Put $X = \Lambda_{01} e^{\mathrm{i}\varphi} r^{-1/\alpha}$ and $k = (1+\alpha)\xi - \beta(1-\beta)$. In the generic chart the leading off-diagonal ($\ell = \pm 1$) data are*

$$F_{\mathrm{off}} = 2\mathrm{Re}\, X, \qquad J_\varphi = (1-2\beta)\, 2\mathrm{Re}\, X, \qquad J_r = -\frac{2(1-2\beta)}{\alpha r}\mathrm{Im}\, X,$$

$$T^t{}_t = T^z{}_z = o(r^{-2-1/\alpha}), \qquad T^r{}_r = -T^\varphi{}_\varphi = -\frac{4k}{\alpha^2 r^2}\mathrm{Re}\, X,$$

$$T^r{}_\varphi = -\frac{4k}{\alpha r}\mathrm{Im}\, X, \quad T^\varphi{}_r = -\frac{4k}{\alpha^3 r^3}\mathrm{Im}\, X.$$

This tensor is traceless, conserved and equal to $(6\xi-1)\mathrm{Box}\, F_{\mathrm{off}}$ *in trace (*F_{off} *is harmonic), and J is conserved; all residuals are exactly* 0 *(sympy, logs/v3_u2_sym.json). The radial current* $J_r \neq 0$ *unless* $\beta = \frac{1}{2}$.

Proof. (i) At large p, $C_{0,-1} = -2c_0(\nu_0)c_0(\nu_{-1})\hat{e}_{\nu_0}\hat{e}_{\nu_{-1}}H_{0,-1}p^{-1/\alpha}(1+O(p^{-2\nu_{\min}}))$. Since $\nu_0+\nu_{-1}>1$, the product of the leading small-argument terms $K_\nu(x) \simeq \frac{\Gamma(\nu)}{2}(x/2)^{-\nu}$ gives a convergent p-integral, which is the $r^{-1/\alpha}$ term. The Mellin convolution of $C_{0,-1}$ with $K_{\nu_0}K_{\nu_{-1}}$ then produces the next term from the pole of the large-p expansion at exponent $1/\alpha$, with coefficient $M(2-1/\alpha)$ (finite since $1/\alpha \notin \mathbb{Z}$). (ii) is the same computation with $C_{0,-1} \simeq -c_0(\nu_{-1})(\hat{e}_{\nu_{-1}}\overline{w}_{-1}/\hat{e}_{\nu_0}\overline{w}_0)p^{\nu_0-\nu_{-1}}$, where all Gamma arguments of $M(2+\nu_0-\nu_{-1})$ are positive. (iii) The leading kernel is the separable harmonic bisolution $\Lambda_{01}U_0(x)\overline{U_{-1}(x')}$ + c.c., $U_j = r^{-\nu_j}e^{\mathrm{i}q_j\varphi}$, to which the point-split stress and current operators are applied; terms with p^2 (from t,z derivatives) are of order $r^{-4+1/\alpha} = o(r^{-2-1/\alpha})$.

Theorem (Extension of the off-diagonal sector). *Let* $\mathscr{N}_{\mathrm{res}}$ *be the closed countable set of* (α,β) *for which an exponent of the asymptotic expansion of the off-diagonal kernel produces an* ε^0 *term in a Ward boundary flux. For* $(\alpha,\beta) \notin \mathscr{N}_{\mathrm{res}}$*:*

1. $F_{\mathrm{off}} \in L^1_{\mathrm{loc}}$ *(scaling degree* $1/\alpha < 2$*): unique extension, no residue, no contact;*

2. J_{off} *(degree* $1+1/\alpha \in (2,3)$*) and* T_{off} *(degree* $2+1/\alpha \in (3,4)$*) have unique finite-part extensions of the same scaling degree modulo* $c^i\delta_{\mathscr{W}}$ *and* $c^\mu{}_\nu{}^k\partial_k\delta_{\mathscr{W}}$, $c\delta_{\mathscr{W}}$*; the finite parts satisfy conservation and the trace identity exactly (fluxes are pure non-integer powers);*

3. *conservation of J kills* c^i*; conservation of* $T^i{}_j$ *forces* c_{ijk} *symmetric in ij and antisymmetric in ik, hence* 0*; the trace identity with* F_{off} *contact-free kills* $\delta^A_B v^k\partial_k\delta_{\mathscr{W}}$*; rotation equivariance (joint rotation of x and of the phase of*

$H_{0,-1}$) excludes $\ell \neq \pm 1$ contacts. Hence the off-diagonal sector has a unique Ward-compatible extension: it carries no residue and no contact. All apex freedom of a $U(2)$ state lies in its $\ell = 0$ sector, which is the three-parameter space (c, τ, w) of Theorem with the coefficients of C_{UV}. On $\mathscr{N}_{\mathrm{res}}$ logarithmic terms may occur; this case is not analysed.

Proof. The leading exponents are non-integer because $1/\alpha \in (1, 2)$. Pairing with ∂f and moving the derivative gives boundary fluxes $\oint T^{r}_{j} f \varepsilon \mathrm{d}\varphi$ that expand in pure powers ε^{γ}, $\gamma \neq 0$ off $\mathscr{N}_{\mathrm{res}}$; the finite part discards them. The contact algebra is the standard one.

Numbers (logs/v3_u2_.json, 20-digit Krein quadrature). $H = \begin{pmatrix} 1 & 0.5e^{0.3\mathrm{i}} \\ 0.5e^{-0.3\mathrm{i}} & 0.7 \end{pmatrix}$.

• $(\alpha, \beta) = (0.8, 0.3)$, $\nu = (0.375, 0.875)$: $\kappa_{01} = -0.27523 - 0.08514\mathrm{i}$. At $r = 10^{-2}, \ldots, 10^{-6}$,

$R_{0,-1}/(\Lambda_{01} r^{-1/\alpha}) = 0.707, 0.886, 0.960, 0.987, 0.9956$ (error $\propto r^{2-2/\alpha}$), imaginary parts $< 10^{-21}$;

$(R_{0,-1} - \Lambda r^{-1/\alpha})/(D' r^{-2+1/\alpha}) \to 0.964$; $\ell = 0$ part over $(\mathrm{E}_0, \mathrm{E}_0)$: 0.979 (error $\propto r^{2-2\nu_{-1}}$).

• $(0.9, 0.45)$, $\nu = (0.5, 0.611)$: two-term ratio $0.957, 0.9917, 0.99854, 0.99972, 0.99990$; $\ell = 0$ ratio 0.99994.

• Boundary chart, $w = (\cos 0.6, \sin 0.6\, e^{0.4\mathrm{i}})$: leading ratio 0.976 at $r = 10^{-6}$ in both cases, and the $\ell = 0$ part tends to E_0 in the larger-ν mode (0.979, 0.994).

Outcome for item 1. Computed: UV-limit theorem, closed-form leading asymptotics of all one-point functions (in terms of the single constant $\kappa_{01}(H)$, plus the local D'), unique off-diagonal extension. The resonant set $\mathscr{N}_{\mathrm{res}}$ is **open**.

Core-model selection revisited: curvature coupling decides (version 3, item 2)

Theorem assumed a self-similar profile $r^{-2} G_{\mathrm{c}}(r/R)$ with power-law approach to s. We now test this assumption on an explicit physical core. In the *flower-pot* model [37, 38], the interior $\rho < R$ is a flat disk, the exterior is the cone, and the curvature is concentrated on the ring $\rho = R$, with $\int R \, \mathrm{d}A = 4\pi(1 - \alpha)$. Take $\beta = 0$, a real field ($n = 1$), $m = 0$ and coupling ξ. Only the $n = 0$ partial wave feels the core at leading order. Its exterior solution is

$J_0(\lambda r) + B(\lambda R)$ (irregular part), and in Euclidean momentum $u = pR$ the matching at the ring gives

$$B(u) = -\frac{uI_1(u/\alpha) - \Lambda I_0(u/\alpha)}{-uK_1(u/\alpha) - \Lambda K_0(u/\alpha)}, \qquad \Lambda = \frac{uI_1(u)}{I_0(u)} + 2\xi(1-\alpha),$$

where the jump $2\xi(1-\alpha)$ is the ξR ring δ (code/v3_core.py). Let $Q(R)$ be the $n = 0$ contribution of the core to the apex coefficient c, normalized as in the code (cutoff $\Lambda_{\mathrm{UV}} = 1$).

Theorem (Three regimes of the core flow).

1. *$\xi = 0$: $B(u) = O(u^2)$, the hypothesis of Theorem holds, and $Q(R)$ converges to a finite, α- and shape-dependent constant (at $\alpha = 0.7$, $Q \to 0.0075167478977544$).*

2. *$\xi > 0$: $B(u) = -1/\log(\kappa/u) + O(\log^{-2})$. The boundary coefficient tends to zero, only logarithmically, and the operator limit is Friedrichs. That is not a proof that the exterior state converges. The apex contact diverges doubly-logarithmically:*

$$Q(R) = -\frac{1}{4\pi}\log\log\frac{1}{R} + O(1).$$

No choice of L absorbs this, because $\sigma_\alpha \mathrm{slog}(R/L)$ is single-logarithmic. Hence the zero-thickness limit selects no point of the line. Before the limit, the core is equivalent to an L_L-type logarithmic boundary condition with running strength. This is the ``fitted strength of logarithmic divergence'' found by Allen–Kay–Ottewill [38, 39].

3. *$\xi < 0$: the ring potential ξR is attractive with $\int \xi R \, \mathrm{d}A < 0$. A two-dimensional Schrödinger operator with negative integrated potential has a bound state [41], so at $m = 0$ there is a negative mode with $\lambda^2 \sim -R^{-2}e^{-c/|\xi|}$. There is no ground state, and the core does not define a Hadamard vacuum of the massless theory.*

Proof. (i), (ii): small-u expansion of the matching ($I_1(u)/I_0(u) \sim u/2$; for $\xi > 0$, $\Lambda \to 2\xi(1-\alpha) > 0$, and $-\Lambda K_0(u/\alpha) \sim \Lambda \log u$ dominates the denominator). Inserting $B \sim -1/\log(1/u)$ into the p-integral for Q gives $\int^{1/R} \mathrm{d}p/(p\log(1/(pR)))$-type terms, hence log log. The coefficient $-1/(4\pi)$ is the normalization of the $n = 0$ term. (iii): Simon's theorem applied to the transverse operator $-\Delta + \xi R$.

Numbers (logs/v3_core_xi*.json, $\alpha = 0.7$, $R = 10^{-2}, 10^{-4}, \ldots, 10^{-32}$). For $\xi = 0$, successive differences in Q are

8×10^{-7}, 8×10^{-11}, 10^{-18}, 0, and $B(10^{-3}) = 3.06\times10^{-7} \approx 0.306u^2$. For $\xi = 0.1, \frac{1}{6}, 0.3$, $B(u)\log(1/u)$ at $u = 10^{-48}$ is $-0.906, -0.942, -0.968$, tending to -1 with $O(1/\log)$ corrections. The slopes of Q against $\log\log(1/R)$ increase monotonically towards the predicted $-1/(4\pi) = -0.0796$: $\xi = 0.1$: $-0.028, -0.041, -0.054, -0.064$; $\xi = \frac{1}{6}$: $-0.037, -0.050, -0.061, -0.069$; $\xi = 0.3$: $-0.047, -0.059, -0.068, -0.073$. Convergence is $O(1/\log\log)$, as expected. It is consistent with the prediction but does not prove it numerically.

Proposition (The $(\xi - \frac{1}{6})R$ delta ambiguity leaves the dilation orbit). *Renormalize φ^2 with the Hadamard parametrix at scale μ, in the convention of Décanini–Folacci [42] with $v_0 = \frac{1}{2}(m^2 + (\xi - \frac{1}{6})R)$. In any core model, the zero-thickness limit of the curvature term is $R \to 4\pi(1-\alpha)\delta_{\mathscr{W}}$. A change $\mu \to \mu'$ then shifts the field-square contact alone:*

$$\delta c = -\frac{n(\xi - \frac{1}{6})(1-\alpha)}{2\pi}\log\frac{\mu'}{\mu}, \qquad \delta\tau = \delta w = 0.$$

For $\xi \neq \frac{1}{6}$ this is not a dilation (a dilation moves (c, τ, w) along $(s, a\text{-slope}, j)$ jointly), so core-model predictions of c depend on the extra scale μ. The ξR coupling does not fix the contact: it introduces a second, independent source for it. At $\xi = \frac{1}{6}$ the ambiguity vanishes.

Proof. $G_H \ni v_0\log(\mu^2\sigma)/(8\pi^2)$, so $\delta\langle\varphi^2\rangle = -(\xi - \frac{1}{6})R\log(\mu'/\mu)/(8\pi^2)$ per component. Insert $R \to 4\pi(1-\alpha)\delta_{\mathscr{W}}$. The stress ambiguity at $m = 0$ is ${}^{(1)}H_{\mu\nu}$, which is quadratic in curvature and not defined at the apex, so it is excluded from the massless class. There are no other terms.

Outcome for item 2. Core dependence proved. For $\xi = 0$ the limit exists and is model-dependent (it is a point of the line). For $\xi > 0$ the limit of c does not exist (log log), and for $\xi < 0$ there is no vacuum. For $\xi \neq \frac{1}{6}$ the μ -ambiguity moves c off the dilation orbit. **Open**: at $\xi = \frac{1}{6}$, whether the triple (c, τ, w) of every core lies on one dilation orbit. This needs the stress-tensor core computation (the τ component), which was not done. Note, though, that at $\xi = \frac{1}{6} > 0$ regime (ii) already rules out a limit of the contact at $\beta = 0$. The Khusnutdinov–Bordag [40] ballpoint-pen model was not computed. The correct identifier is arXiv:gr-qc/9810066, Phys. Rev. D **59** (1999)

064017, ``Ground state energy of a massive scalar field in the background of a cosmic string of finite thickness". That paper uses the Gott–Hiscock metric. In the naming already used by Allen–Ottewill for the flower-pot, the Gott–Hiscock constant-density spherical-cap interior is the ballpoint-pen core, not the flat-disk flower-pot computed in this section. The label therefore stays, and the ground-state energy is still not computed here. The number gr-qc/9810049 previously quoted resolves to Cho–Vilenkin, ``Gravitational field of vacuumless defects", and is the wrong paper.

Worldsheet restriction, boundary fields and the apex Wick square (version 3, item 3)

This section replaces Remark . Throughout, $y = (t, z)$ are worldsheet coordinates and $\Delta y = y - y'$.

Lemma (Transverse Sobolev order). *Near $r = 0$ a Friedrichs mode is $r^{\nu} e^{\mathrm{i}q\varphi}$ and an E_0-type (singular) mode is $r^{-\nu} e^{\mathrm{i}q\varphi}$. Cut off near 0, $r^{\pm\nu} e^{\mathrm{i}q\varphi} \in H^s(\mathbb{R}^2)$ iff $s < 1 \pm \nu$. Hence the Friedrichs two-point function is, in each variable, of transverse order $H^{1+\nu_{\min}-}$ ($\beta \neq 0$) or smooth plus $H^{1+1/\alpha-}$ ($\beta = 0$), above the trace threshold $s > 1$ for restriction to a point; singular modes are of order $H^{1-\nu-}$, below it.*

Theorem (Restriction $\varphi|_{\mathscr{W}}$).

1. *$\beta = 0$, Friedrichs: only the $n = 0$ mode survives at $r = 0$, and $G_F((y,x),(y',0)) = \dfrac{1}{4\pi^2\alpha\,(r^2 + |\Delta y|_E^2)}$. Hence $\varphi|_{\mathscr{W}}$ exists as an operator-valued distribution on the Friedrichs Fock space and is the $1+1$-dimensional generalized free field of scaling dimension 1 with two-point function α^{-1} times the flat one restricted to a timelike 2-plane, $\langle \varphi|_{\mathscr{W}}(y)\varphi|_{\mathscr{W}}(y')\rangle = (4\pi^2\alpha\,(-(\Delta t - \mathrm{i}0)^2 + \Delta z^2))^{-1}$; $\|\varphi(h \otimes \delta_0)\Omega\|^2 = \frac{1}{2\alpha}\int \mathrm{d}p_z \int_{|p_z|}^{\infty} \mathrm{d}\omega\, |\widehat{h}(\omega, p_z)|^2 < \infty$ (normalization of the flat case times α^{-1}).*

2. *$\beta \neq 0$, Friedrichs: $\varphi|_{\mathscr{W}} = 0$ (every mode vanishes like $r^{\nu_{\min}}$).*

3. *Singular modes: $\varphi|_{\mathscr{W}}$ does not exist (Lemma); instead the boundary value $a_j(y) = \lim_{r \to 0} r^{\nu_j} \frac{1}{2\pi}\int_0^{2\pi} e^{-\mathrm{i}q_j\varphi}\varphi\,\mathrm{d}\varphi$ exists for every extension with $a_j \equiv 0$.*

Theorem (Boundary fields are worldsheet generalized free fields). *For a nonnegative extension in the chart H*

(Krein matrix C), the boundary fields a_j exist in the Fock space and

$$\langle a_j(y)a_k^\dagger(y')\rangle_E = \frac{\Gamma(\nu_j)\Gamma(\nu_k)2^{\nu_j+\nu_k}}{16\pi^2\alpha}\int_0^\infty p\,\mathrm{d}p\,J_0(p|\Delta y|)\,p^{-\nu_j-\nu_k}C_{jk}(p).$$

1. E_0 *in mode j:* $\langle a_j a_j^\dagger\rangle_E = \frac{|\Delta y|^{2\nu_j-2}}{4\pi^2\alpha}$*, the two-dimensional generalized free field of dimension* $\Delta_- = 1-\nu_j$*.*

2. E_λ*: the momentum-space two-point function is* $\propto (p^{2\nu}+M_\lambda^{2\nu})^{-1}$*, a Stieltjes function of* p^2*, so a has the Källén–Lehmann measure* $\mu(m^2)\propto m^{2\nu}\sin(\pi\nu)/|m^{2\nu}e^{i\pi\nu}+M_\lambda^{2\nu}|^2>0$*. It interpolates between dimension* $1-\nu$ *at* $|\Delta y|\ll M_\lambda^{-1}$ *and* $1+\nu$ *at* $|\Delta y|\gg M_\lambda^{-1}$*: the worldsheet analogue of a double-trace flow between the two boundary dimensions* $\Delta_\pm = 1\pm\nu$*.*

3. *Off-diagonal* $U(2)$*:* (a_0, a_{-1}) *is a two-component generalized free field with positive matrix spectral density* $\propto p^{-\nu_j-\nu_k}C_{jk}(p)$ *(positive because* $C>0$ *for* $H\geq 0$*).*

Proof. Theorem (i): the $n=0$ Friedrichs mode is $J_0(\lambda r)$ and $\int_0^\infty \lambda J_0(\lambda r)K_0(\lambda D)\mathrm{d}\lambda = (r^2+D^2)^{-1}$; the norm follows from the spectral representation of the flat case, where $\mathrm{d}^2p_\perp/(2\omega) = \pi\,\mathrm{d}\omega$ at fixed p_z. Theorem : insert $K_\nu(pr)\simeq\frac{\Gamma(\nu)}{2}(pr/2)^{-\nu}$ in the Krein formula; the Friedrichs part is $O((rr')^\nu)$ and drops out. For (i), $c_0\Gamma(\nu)^2 4^\nu/4\cdot\int_0^\infty p^{1-2\nu}J_0(pD)\mathrm{d}p = c_0\Gamma(\nu)\Gamma(1-\nu)/2\cdot D^{2\nu-2} = D^{2\nu-2}$. (ii) $1/(x^\nu+c)$ is Stieltjes for $0<\nu<1$, $c>0$; the large-D term comes from $-p^{2\nu}/M^{4\nu}$ and $\int_0^\infty p^{1+2\nu}J_0(pD)\mathrm{d}p = 2^{1+2\nu}\Gamma(1+\nu)/(\Gamma(-\nu)D^{2+2\nu})$.

Theorem (Wick square smeared on the worldsheet). *Let* ω *be the Friedrichs state.*

1. *For every f in* $C_c^\infty(\overline{M})$ *and every apex-supported* $h\otimes\delta_{\mathscr{W}}$*,* $h\in C_c^\infty(\mathbb{R}^2)$*, the normal-ordered square* $:\varphi^2:_\omega$ *is a densely defined Fock operator. Indeed* $\|:\varphi^2:(f)\Omega\|^2 = 2\iint \mathrm{d}\omega\,\mathrm{d}\omega'\langle\hat{f}_{\omega+\omega'},(E_\omega\otimes E_{\omega'})\hat{f}_{\omega+\omega'}\rangle$*, and the diagonal spectral function satisfies* $E_\omega(x,x)\leq C\omega$ *uniformly up to the apex.*

2. *For* $\beta=0$*,* $:\varphi^2:(h\otimes\delta_{\mathscr{W}}) = :(\varphi|_{\mathscr{W}})^2:(h)$*, a dimension-2 generalized free field with two-point function* $2(4\pi^2\alpha\,\Delta y^2)^{-2}$*; for* $\beta\neq 0$ *it is* 0*.*

3. *The locally covariant Wick square* $\varphi^2 = :\varphi^2:_\omega + \langle\varphi^2\rangle_\omega\mathbb{1}$ *cannot be smeared with* $h\otimes\delta_{\mathscr{W}}$*: its c-number part*

$sU_L + c\delta_{\mathscr{W}}$ has scaling degree 2 *equal to the codimension and no restriction to $\mathscr{W}$. The admissible c-number parts on $C_c^\infty(\overline{M})$ are exactly $\widehat{F}_L + c\delta_{\mathscr{W}}$.*

4. *For extensions with a singular mode, $E_\omega(x,x) \sim r^{-2\nu}$ and $:\varphi^2:(h\otimes\delta_{\mathscr{W}})$ diverges; the correct apex object is the boundary Wick square $:a_j a_j^\dagger:$, of dimension $2-2\nu_j$, which exists.*

Numbers (logs/v3_wick.json). $(\alpha,\beta) = (0.7, 0.3)$: $(rr')^\nu \delta G/(|\Delta y|^{2\nu-2}/4\pi^2\alpha)$ at $r = 10^{-1},\dots,10^{-5}$ tends to 1 ($1 - 3\times10^{-9}$ at 10^{-5}); the Krein identity in (i) holds to 3×10^{-26}. For E_λ ($M = 1$) the direct two-point function and its Källén–Lehmann representation agree to $\le 4\times10^{-10}$ at $|\Delta y| = 10^{-2},\dots,10^{2}$; the UV ratio is 0.94 at 10^{-2} and the IR ratio 0.98 at 10^{2}. The mode integral for $G_F(x,\text{apex})$ reproduces $1/(4\pi^2\alpha(r^2+D^2))$ to all digits at $\alpha = 0.7, 0.5$, and the $\alpha = \frac{1}{2}$ image sum agrees.

Outcome for item 3. Computed: $\varphi|_{\mathscr{W}}$ (exists for $\beta = 0$, vanishes for $\beta \neq 0$, excluded with proof for singular modes), boundary fields a_j with explicit 2D two-point functions, normal-ordered Wick square on $\mathscr{W}$. **Excluded with proof**: the c-number part of the covariant Wick square on $\mathscr{W}$. The uniform bound $E_\omega(x,x) \le C\omega$ is used in the form of the heat-kernel bound already cited for Theorem .

Massless Dirac field on the cone with flux (version 3, item 4)

Let ψ be a massless Dirac field of charge e on $\mathbb{R}^{1,1}\times C_\alpha$ with Aharonov–Bohm flux β, $0 \le \beta \le \frac{1}{2}$ (the case $\beta \in [\frac{1}{2}, 1)$ follows from charge conjugation, $\beta \mapsto 1-\beta$). The squared Dirac operator acts on the two chirality-σ_3 components as a charged Laplacian with spin connection, and so has angular orders $|n+\beta_s|/\alpha$ with

$$\beta_\pm = \beta + \frac{1\mp\alpha}{2} \qquad \text{(branch B)}, \qquad \text{or} \qquad \beta_\pm = \beta - \frac{1\pm\alpha}{2} \qquad \text{(branch A)}.$$

For $\beta \le (1-\alpha)/2$ every partial wave has a unique regular solution, so the regular choice is the only one [36]. For $|\beta - \frac{1}{2}| < \frac{\alpha}{2}$ one partial wave has a two-parameter family, with two scale-invariant members, the branches A and B. Coincidence limits of the squared operator are those of charged scalars with $\xi = \frac{1}{4}$ and $n = 2$, taken with a relative sign fixed below.

Theorem (Closed forms). *Write* $T^{\mu}{}_{\nu} = h_D\, r^{-4}\mathrm{diag}\,(1,1,-3,1)$ *in* (t,r,φ,z) *and* $J_{\hat{\varphi}} = j_D/(\alpha r^3)$ *(orthonormal component). Then* $\langle\bar{\psi}\psi\rangle = 0$, $J_t = J_r = J_z = 0$, *and*

$$h_D^{\mathrm{B}} = -\frac{(\alpha^2-1)(17\alpha^2+7)+120\beta^2(\alpha^2-2\beta^2+1)}{2880\pi^2\alpha^4}, \qquad j_D^{\mathrm{B}} = \frac{\beta\,(1+3\alpha^2-4\beta^2)}{12\pi^2\alpha^2},$$

$$h_D^{\mathrm{A}}(\beta) = h_D^{\mathrm{B}}(1-\beta), \qquad j_D^{\mathrm{A}}(\beta) = -j_D^{\mathrm{B}}(1-\beta).$$

Branch B is the regular state for $\beta \le (1-\alpha)/2$ *and is the state of Bezerra de Mello–Saharian [33] in the window. The trace vanishes, and the tensor is conserved. At* $\beta = 0$, $h_D = (\nu^2-1)(7\nu^2+17)/(2880\pi^2)$, $\nu = 1/\alpha$ *[35, 26].*

Proof. For each component, $\langle T\rangle$ and J are sums of the charged-scalar polynomials of Theorem at $\beta_\pm$ with $\xi = \frac{1}{4}$: $h_D = -[\tilde{a}(\beta_+) + \tilde{a}(\beta_-)]$, $\tilde{a} = h + 4n(\xi - \frac{1}{6})s$, and $j_D = j(\beta_+) + j(\beta_-) - \alpha(P(\beta_+) - P(\beta_-))/(24\pi^2)$, $P = \alpha^{-2}(1-6\beta(1-\beta)) - 1$. The overall minus sign is Fermi statistics; the last term is the spin-magnetization current. sympy (code/v3_dirac.py) gives the stated polynomials. The relative sign of the spin term is fixed by comparison with the literature (below); the other sign gives $-\beta(9\alpha^2+4\beta^2-1)/(12\pi^2\alpha^2)$, which disagrees with the literature and with the $\beta = (1-\alpha)/2$ closed form.

Checks and literature (logs/v3_dirac*.json).

• $\beta = 0$: h_D minus Linet's eq. (39) is 0 symbolically.

• Current versus the massless limit of Bezerra de Mello–Saharian [33] eq. (3.30) (series plus integral representation, evaluated independently at 30 digits). Outside the window, at $(\alpha,\beta) = (0.8,0.05),(0.6,0.1),(0.45,0.15),(0.7,0.12),(0.35,0.2)$, the agreement is $\le 4\times10^{-18}$ relative. In the window, at $(0.8,0.2),(0.8,0.3),(0.6,0.3),(0.6,0.45),(0.45,0.4),(0.9,0.1)$, branch B agrees to $\le 3\times10^{-16}$ and branch A does not. At $\beta = (1-\alpha)/2$ the difference from their eq. (3.35) is 0 symbolically.

• Energy, independent check. A heat-kernel mode sum over the Bezerra de Mello–Saharian mode orders ($\sum I_{q(n\pm\beta+\frac{1}{2})\mp\frac{1}{2}}$, scipy) is normalized only at $\beta = 0$; the normalization constant is $K\pi^2 = -0.25000000000$ at both $\alpha = 0.8$ and 0.6 (it must be α-independent). It then reproduces h_D^{B} at $\beta = 0.05, 0.1, 0.25, 0.4$ to $\le 1.1\times10^{-10}$

relative.

• **Disclosed disagreement.** Our transcription of Spinelly–Bezerra de Mello [34] (hep-th/0306103) eq. (59) gives the bracket $(\alpha^2-1)(17\alpha^2+7)+120\beta^2(\alpha^2-2\beta^2-1)$, and our result has $(\alpha^2-2\beta^2+1)$. The two agree at $\beta=0$ and differ for every $\beta\neq 0$; at $(0.8, 0.25)$ ours is -4.231×10^{-4} and theirs $+8.653\times10^{-4}$. Two independent methods support our sign (polynomial decomposition, and the mode sum). Our result is also continuous with the Bezerra de Mello–Saharian current, which fixes the spin-term sign. We have not resolved whether the difference is a misprint, a different flux convention, or our transcription. We make no claim that their paper is in error.

Theorem (Apex extension for the Dirac field). *For either branch, the data (T, J) are exactly homogeneous, with $(s, a, b, j) = (0, h_D, h_D, j_D)$. The extension theorem (Theorem (iii)) applies verbatim to the stress and the current, with $T^{\mu}{}_{\mu}=0$ imposed as the Ward identity:*

$$(E+4)\widehat{T}^{A}{}_{B}=\frac{h_D}{4}\sigma_{\alpha}\,\delta^{A}_{B}\Delta\delta_{\mathscr{W}},\qquad (E+4)\widehat{T}^{i}{}_{j}=\frac{h_D}{2}\sigma_{\alpha}K_{ij}\delta_{\mathscr{W}},\qquad (E+3)\widehat{J}^{i}=-\frac{j_D}{2\alpha^2}\sigma_{\alpha}\,\varepsilon^{i}{}_{k}\partial_k\delta_{\mathscr{W}}.$$

The contact space is (τ, w), two real parameters: there is no c, because there is no scalar density and the trace identity is homogeneous. When $\beta=0$, $j_D=0$; charge conjugation then forbids w, and the space is $\{\tau\}$. The generator line of Theorem becomes the line $L\mapsto(\tau(L), w(L))$ with slope (h_D, j_D). The slope is never zero. At $\beta=0$, $h_D=(v^2-1)(7v^2+17)/(2880\pi^2)>0$. For $0<\beta\le\frac{1}{2}$, $j_D>0$ because $1+3\alpha^2-4\beta^2>0$. Note that h_D itself changes sign, for example $h_D<0$ at $(0.8, 0.25)$. Hence the no-go theorem (Theorem) holds for the Dirac field: no canonical dilation-invariant point exists.

Caveat. The trace anomaly of the Dirac field is quadratic in curvature and has no meaning at the apex. Imposing $T^{\mu}{}_{\mu}=0$ as a distributional Ward identity is a declared choice; it is not derived.

Outcome for item 4. Computed: closed forms for $\langle\bar{\psi}\psi\rangle$, $\langle T\rangle$ and $\langle J\rangle$ for the regular state and both window branches; literature agreement as listed, with one disclosed, unresolved disagreement; apex residues and the contact space. General (non-scale-invariant) window extensions are **open**. They would follow by combining Section with the spinor Krein formula, which is not written here.

First-order lambda phi4: the tadpole and the fate of the contact (version 3, item 5)

Take the real field ($\beta = 0$, $n = 1$) with Euclidean interaction $\frac{\lambda}{4!}\varphi^4$ and the Friedrichs state. To first order,

$$\delta_\lambda\langle\varphi^2(x)\rangle = -\frac{\lambda}{2}\int \mathrm{d}\Omega_y\, G_F(x,y)^2\,\langle\varphi^2(y)\rangle_{\mathrm{ext}}, \qquad \langle\varphi^2\rangle_{\mathrm{ext}} = sU_L + c\,\delta_{\mathscr{W}},$$

where the composite operator φ^2 is renormalized at scale μ at $y = x$.

Proposition (First-order apex dependence). *For $r > 0$,*

$$\delta_\lambda\langle\varphi^2\rangle = -\frac{\lambda\, c}{32\pi^3\alpha^2 r^2} - \frac{\lambda s}{16\pi^2\alpha}\frac{\log(r/L)}{r^2} - \frac{\lambda s}{16\pi^2}\frac{\log(\mu r)}{r^2} + \frac{\lambda s\, A_\alpha}{r^2},$$

with a scheme-dependent constant A_α (not computed). In particular:

1. *the free apex contact c is no longer a contact: at first order it changes the r^{-2} coefficient off the apex. The combination $c + \sigma_\alpha s\log(r/L)$ is dilation invariant, so the contact freedom becomes a physical worldsheet coupling, equivalent to a term $\frac{\lambda c}{2}\int_{\mathscr{W}}\varphi^2$;*

2. *for $\beta \neq 0$ (charged field, Friedrichs) $G_F(x,\cdot)|_{\mathscr{W}} = 0$, so neither c nor L enters at first order;*

3. *at second order the worldsheet coupling needs $\int_{\mathscr{W}} G_F(y,y')G_F(y',x)\mathrm{d}^2y'$ with $G_F(y,y') = 1/(4\pi^2\alpha\,\Delta y^2)$ on $\mathscr{W}$, which diverges logarithmically. A δ-potential on the apex of a two-dimensional transverse space must be renormalized; its renormalized form is the L_L family, which at $m = 0$ has a bound state (I-B-VP-SAE thm:spec(ii)). This is the precise **obstruction** beyond first order.*

Proof. $G_F(x,(y,0)) = 1/(4\pi^2\alpha(r^2 + |\Delta y|^2))$ (Theorem), so $\int_{\mathscr{W}} G_F^2 = \int \mathrm{d}^2\Delta\,(4\pi^2\alpha)^{-2}(r^2+\Delta^2)^{-2} = (16\pi^3\alpha^2 r^2)^{-1}$, which gives the *c* term. Near the apex $G_F(x,y)^2 = G_F(x,(y,0))^2(1 + O(r_y^{1/\alpha}))$, so the finite part at scale *L* contributes $\sigma_\alpha\log(r/L)$ times the same integral. Near $y = x$, $G_F^2 \simeq (16\pi^4|x-y|^4)^{-1}$ and $\int_{|x-y|>\mu^{-1}} \frac{\mathrm{d}^4y}{16\pi^4|x-y|^4} = \frac{\log(\mu r)}{8\pi^2} + O(1)$, the one-loop anomalous dimension of φ^2. Dimensional analysis fixes the remaining constant up to A_α.

Numbers. $\int_{\mathscr{W}} G_F^2$ by quadrature over the closed form at $r = 0.37$, $\alpha = 0.7$: ratio 1 to all digits (logs/v3_wick.json).

Outcome for item 5. First order **computed** up to the scheme constant A_α (**open**); beyond first order **open** with the stated obstruction. The scope is the free Friedrichs vacuum perturbed at $O(\lambda)$; no interacting Ward theorem is claimed.

Audit of version 2 and record of version-3 checks (version 3, item 6)

Re-audit of the new results of version 2 (code/v3_audit.py, logs/v3_audit.json).

1. *No-go theorem.* The hypothesis $(s, h, j) \neq 0$ of Theorem holds for *every* Friedrichs datum, not only generically. $s = 0$ requires $\beta = \frac{1}{2} \mp \frac{1}{6}\sqrt{6\alpha^2 + 3}$, and there $j^2(12\pi^2\alpha^2)^2 = \frac{(1-\alpha^2)^2(1+2\alpha^2)}{108} > 0$. This is a sharpening; there is no error.

2. E_0 *positivity.* On a 51×49 grid of $[0, 1] \times (0, 1)$, the minima of the three field-square numerators of Theorem are $0.0396, 0.0396, 10.54$, all > 0. Confirmed.

3. *Mass residue.* Re-derived symbolically as $-\frac{1}{2}nm^2 s\, \sigma_\alpha \delta_{\mathscr{W}}$ from $(\mathbb{E} + 4)(m^2\widehat{F}) = m^2(\mathbb{E} + 2)\widehat{F}$. Confirmed.

4. *Line family.* $\Delta_{\mathrm{cone}}(r^{-2}\cos\varphi) = (4 - \alpha^{-2})r^{-4}\cos\varphi$. Confirmed.

5. *Thermal contact space.* A re-run of code/class_extra.py (logs/v3_class_extra_rerun.log) gives dimensions $4, 3, 5$ for the three classes, as in version 2. Confirmed.

Corrections to version 2.

1. *Theorem (``core models realize every point of the line'').* The theorem is correct as stated for self-similar profiles with power-law tails. However, its interpretation in version 2 as a statement about *physical* cores is wrong in general. For $\beta = 0$ and $\xi > 0$, the flower-pot core has a 1/log tail, and c diverges like $\log\log$ (Theorem). For $\xi \neq \frac{1}{6}$, the μ-ambiguity of Proposition leaves the line. Only $\xi = 0$ cores fall under Theorem .

2. *Ledger, non-diagonal $U(2)$.* The ``Open'' entry is closed. The hypothesis of Proposition is proved (Theorem), and the off-diagonal sector is computed (Theorems ,).

3. *Remark* . The stated obstruction (``no wave-front control at $\mathscr{W} \times \mathscr{W}$'') is superseded for apex-supported smearings by Theorems –. The control needed is only the transverse Sobolev order of Lemma .

4. *Abstract of version 2.* The abstract applies Theorem to ``every rotation-invariant stationary state". The theorem's W1–W3 part also used boost invariance. Thermal states are not boost-invariant; for them, version 2 proved the result separately (Theorem ). The wording should have said ``with the thermal case treated separately". No result changes. No error was found in the thermal dimension-5 contact space, the massive residue, or the no-go theorem.

Version-3 check record. toprule ID Check Result

midrule endhead V1 $U(2)$ leading off-diagonal asymptotics ratio $\to 1$ (0.9956 at $r = 10^{-6}$); two-term 0.99990

V2 $U(2)$ symbolic Ward identities all residuals 0

V3 E_0 boundary-field limit, Krein identity $1 - 3 \times 10^{-9}$; 3×10^{-26}

V4 E_λ boundary field vs. Källén–Lehmann $\le 4 \times 10^{-10}$

V5 $G_F(x, \mathrm{apex})$ closed form; $\int_{\mathscr{W}} G_F^2$ exact to working precision

V6 Dirac energy at $\beta = 0$ vs. Linet (39) 0 (symbolic)

V7 Dirac current vs. BdM–S (3.30), 11 points $\le 3 \times 10^{-16}$ (branch B)

V8 Dirac energy, heat-kernel mode sum vs. closed form $\le 1.1 \times 10^{-10}$; $K\pi^2 = -\frac{1}{4}$ for both α

V9 Dirac energy vs. Spinelly–BdM (59) **disagree** for $\beta \neq 0$ (disclosed)

V10 Flower-pot core, $\xi = 0$ convergence differences $\to 10^{-18}$

V11 Flower-pot core, $\xi > 0$ log log slope monotone towards $-1/(4\pi)$; not converged

V12 v2 re-audit A1–A5 confirmed

bottomrule

What this completion does and does not provide

Regular-part agreement and supported laboratories

Every test supported in $\{r \geq r_0 > 0\}$ sees exactly the regular one-point functions, for all c, τ, w, L. No extension parameter alters the Friedrichs wave operator, the regular spectral state, or its local Hadamard subtraction.

The supported-transfer theorem of I-B-G concerns scalar retarded and advanced operators, with both the additional-arrival window and the independent retained-shadow reserve [8, thm:transfer, ass:transfer-contract]. The X-A-ST and X-B-ST supported-chart contracts remain separate [11, 12]. Neither Green-operator agreement nor a local developing chart transfers a vacuum covariance or selects apex contacts. No comparison with a flat-completion state is used in the proofs above.

Energy and positivity are additional questions

For a static observer on M, the measured energy density is $-a/r^4$. Its cutoff transverse integral per unit longitudinal length is

$$\int_{\varepsilon < r < R} -\frac{a}{r^4}\, \alpha r \, \mathrm{d}r \, \mathrm{d}\varphi = -\frac{\sigma_\alpha a}{2}\left(\varepsilon^{-2} - R^{-2}\right).$$

It generally diverges. Likewise,

$$\int_{\varepsilon < r < R} F^\circ \, \alpha r \, \mathrm{d}r \, \mathrm{d}\varphi = \sigma_\alpha s \log\left(R/\varepsilon\right).$$

A finite distributional action on smooth compact tests is not a proof of a finite unrenormalized energy per unit length.

If $s \neq 0$ (either sign), no extension of s/r^2 is a signed Radon measure, i.e. a distribution of order zero: such a measure would have locally finite total variation, whereas every neighbourhood of the apex has infinite regular variation $\int |s| r^{-2} \, \mathrm{d}\Omega$ (sharpened in version 2, E4). In particular the constructed finite part is not a positive measure. This does not contradict positivity of the ground-state two-point function on regular tests.

No nonlinear singular geometry is manufactured

The conical metric in the chosen smooth coordinates is

$$g_{ij} = \alpha^2\delta_{ij} + (1-\alpha^2)\frac{x_i x_j}{r^2} \qquad (r > 0).$$

It has no direction-independent value at $x = 0$ when $\alpha \neq 1$. Products such as $g_{ij}\partial_k\delta_{\mathscr{M}}$, a distributional Einstein tensor built from unspecified products, or a singular-connection divergence of arbitrary contacts are not defined by the finite parts above.

Our mixed tensors are ordinary tensor distributions on the *auxiliary smooth completion*. Their regular restrictions are the physical mixed tensors, and their extended Ward identities are those fixed in Definition . A core regularization or nonlinear generalized-function framework would require a separate construction and comparison [5]. No such comparison is asserted.

Finite reporting, coverage, and evidence

The finite supplied-report instrument can install words containing

$$(\alpha, \beta, \xi, L, c, \tau, w, \text{formula identifiers}, \text{test descriptions}, \text{evidence status}).$$

For a calibrated clock basis $\{|\tau_j\rangle\}$, set

$$D_j = \langle\tau_j|_C \otimes I_A.$$

The recorded instrument is

$$\mathscr{M}(X) = \sum_j \text{lvertj}, w_j\rangle\text{langlej}, w_j|_R \otimes D_j X D_j^\dagger.$$

Its Kraus completeness and discard identities are

$$\sum_j D_j^\dagger D_j = I_{CA}, \qquad \mathrm{Tr}_R \circ \mathscr{M}(X) = \mathrm{Tr}_C X,$$

including every untouched-reference extension [7, thm:instrument] [19, 18]. It returns the surviving A, not a

quantum C-register. The words are installed descriptions, not measurements acquiring an unknown continuum distribution.

Coverage is class relative. Every regular-part test annihilates all contacts in Theorem . Thus even the collection of *all* tests supported away from $\mathscr{W}$ cannot select c, τ, w.

Within the declared three-dimensional contact space, a finite family of appropriately chosen apex-jet tests can be injective: the three generators are independent distributions, and linear algebra selects separating test actions. That conclusion uses the proved classification. It does not hold for an unrestricted space of worldsheet-supported distributions. This is the finite-coverage distinction of Oai282-A [20].

Evidence classes. We retain the reporting discipline used in XVI-O1-ST [18]:

\texttt{ANALYTIC} A statement established by the displayed argument and its identified analytical inputs.

\texttt{SUPPLIED} An inherited computational report, not independently rerun.

\texttt{UNCERTIFIED} A numerical enclosure or verification claim without the artifacts and total error bounds required for that interpretation. The theorems of this paper are analytical. Version 2 adds independent symbolic and numerical reruns (Section); the numerical ones are quadrature residuals without total error bounds and are therefore in the above sense, although their scripts and raw logs are retained. The numerical residuals in the regular-part companion papers are not relabeled as checks of these extension identities.

A future regulated numerical report must retain the test, density, smooth structure, component convention, cutoff profile, input encodings, raw subtraction terms, precision, residuals, and remainder bounds. Input error, cutoff error, and quadrature error are separate. A successful regression that retains a known discrepancy does not remove it. Version 2 compiles with pdflatex; numerical execution is recorded as above; no proof-assistant verification is asserted.

Ledger update and conclusion

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Friedrichs regular one-point data Retained; re-audited Unchanged on every regular-part test; coefficients rechecked against I-B-VP-AB (exact).

Apex finite parts (Friedrichs) Constructed Strong distributions on the completion of Assumption ; rate $O(\varepsilon^2)$ checked (N1).

Simultaneous Ward identities Constructed For $\mathfrak{D}$ (diagonal data) and for the Killing–homothety form W1–W3 (all stationary boost-invariant data).

Contact ambiguity Classified Dimension 3 under reduced hypotheses (Proposition); other classes in Table .

Strict normal homogeneity Excluded with proof No extension of sr^{-2}, $s \neq 0$, is homogeneous (Proposition); jointly homogeneous members exist when an intrinsic scale exists (Theorem (vi)).

(a) E_0, $(\mathrm{E}_0, \mathrm{E}_0)$ **Computed** Version 1 at shifted flux; $s > 0$ always (Theorem).

(a) E_λ **Computed** Residues equal the E_0 values and are scale independent; κ-family (Theorem).

(a) $\beta = \frac{1}{2}$ line family **Computed** Isotypic extension; explicit failure of $\mathfrak{D}$; W1–W3 hold (Theorem).

(a) non-diagonal $U(2)$, $H \geq 0$ **Computed** (v3) UV-Lagrangian theorem; leading off-diagonal asymptotics $\propto \kappa_{01}(H) r^{-1/\alpha}$, $J_r \neq 0$; unique off-diagonal extension (Theorems –). Resonant set: **open**.

(b) Thermal Friedrichs **Computed** L^1_{loc} correction; vacuum residues; 5-dimensional contacts; agrees with Guimarães and FPZ (Theorem).

(c) Massive Friedrichs **Computed** Proper-time representation, small-mass expansion, mass residue (Theorem , Proposition).

(d) Canonical apex renormalization **Excluded with proof** in $\mathscr{A}$; a line selected No dilation-covariant point; single-generator line; core constant model dependent (Section). Physical cores (v3): **core dependence proved**;

$\xi > 0$ gives log log divergence, $\xi < 0$ no vacuum, $(\xi - \frac{1}{6})R$ leaves the orbit (Section). $\xi = \frac{1}{6}$ orbit universality: **open**.

(e) Field algebra at the apex Linear algebra **constructed** Local von Neumann algebras are unchanged by the apex (Theorem , under a cited heat-kernel bound).

(e) Wick-square operators, $\varphi|_{\mathscr{W}}$ **Computed** (v3) $\varphi|_{\mathscr{W}}$ exists iff $\beta = 0$ (Friedrichs); boundary fields a_j are 2D generalized free fields; : φ^2 : on $\mathscr{W}$ exists; covariant c-number part: **excluded with proof** (Section).

Massless Dirac with flux **Computed** (v3) Closed forms; Linet, BdM–S agree; Spinelly–BdM (59) disagreement disclosed; contacts (τ, w); general window extensions **open** (Section).

$\lambda\varphi^4$, first order **Computed** up to a scheme constant (v3) c becomes physical $(-\lambda c/32\pi^3\alpha^2 r^2)$; second order: **open** with δ-potential obstruction (Section).

Re-audit of version 2 Done (v3) Theorems confirmed; interpretation of Theorem corrected (Section).

endgroup

The precise V1 advance. The Friedrichs one-point extension question is partially resolved: a concrete completion prescription exists, its direct-cutoff defects are repaired, and its residual contact space is fully classified. The original demand for a uniquely normalized or intrinsically defined apex observable is not thereby resolved. Version 2 extends the resolution to the E_0, (E_0, E_0), E_λ, line-family, thermal and massive states, proves that no canonical point exists in the declared class, and constructs the linear apex field algebra. V2, V3, and V5–V8 of I-B-VP retain their stated scopes; I-B-VP-0's current evaluation remains its separate regular-part achievement [9, 10].

Version 3 closes the non-diagonal $U(2)$ entry and computes worldsheet fields, the Dirac field and the first-order tadpole. It shows that physical cores do not select a point of the contact line. The zero-thickness limit of the apex contact is model-dependent at $\xi = 0$, does not exist for $\xi > 0$, and has no vacuum for $\xi < 0$. That statement is about the contact, not a denial of the operator limit. In the monotone class of I-B-VP-5 the operator limit is Friedrichs. Theorem (ii) records only that the boundary coefficient tends to zero. Strong resolvent convergence is not a state

limit.

Non-sourcing and unchanged programme boundaries. A prescribed distributional extension is not a field equation. No Maxwell equation, semiclassical Einstein equation, backreaction solution, spacetime emergence, or interacting Ward theorem is constructed. The non-sourcing distinctions of I-B and X-A-ST remain unchanged [7, 11].

No external OpenAI catalogue entry is a premise. No characteristic graph coefficient is constructed by this one-point extension. In particular, the existence of technical XV-B manuscripts is not the opening of its distinct downstream quantum-admission exit [15, 19]. O.1 remains wherever it was not independently closed; XVI-N is not discharged; is not promoted [18, 19]. These restrictions do not deny the free-field distributions proved here.

Conclusion. Version 2 shows that the construction is robust: it extends, through three scalar potentials, to every rotation-invariant stationary state with an r^{-2} apex singularity, with residues determined by local apex data. It also shows that it is intrinsically non-canonical: dilations act freely on the contacts.

A declared smooth completion and Ward representative turn the regular r^{-2}, r^{-4}, and circulating-current formulas into a precise extension problem. One generating finite part solves it simultaneously. The raw finite-part defects, their repair, the remaining contact parameters, and the nonremovable dilation residues are explicit. The construction is therefore more than a proposal, but deliberately less than a canonical singular-background quantum theory or a derived source law.

appendix

Boundary calculations for the finite conversions

We give the local transverse calculation; longitudinal integration is passive. Write $\langle\cdot\rangle_{\varphi}$ for an unnormalized angular integral, and use the constant density factor α. All remainder estimates are uniform in the required compact longitudinal set.

For a smooth test,

$$\frac{1}{2\pi}\langle f(\varepsilon\omega)\rangle_{\varphi} = f(0) + \frac{\varepsilon^2}{4}\Delta f(0) + O(\varepsilon^4 p_4(f)).$$

The angular identities used below are

$$\int_0^{2\pi} \omega_i \omega_j \,\mathrm{d}\varphi = \pi\delta_{ij},$$

$$\int_0^{2\pi} \omega_i \omega_j \omega_k \omega_l \,\mathrm{d}\varphi = \frac{\pi}{4}\left(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right).$$

One derivative

For $u = r^{-2}$, integration by parts on $r > \varepsilon$ uses the inner outward normal $-\omega$:

$$\alpha \int_{r>\varepsilon} u\,\partial_i f \,\mathrm{d}^2x = -\alpha \int_{r>\varepsilon} \partial_i u\, f \,\mathrm{d}^2x - \alpha\varepsilon^{-1}\int_0^{2\pi} \omega_i f(\varepsilon\omega)\,\mathrm{d}\varphi.$$

The last expression equals

$$-\frac{\sigma_\alpha}{2}\partial_i f(0) + O(\varepsilon^2 p_3(f)).$$

After including the logarithmic subtraction in U_L, this gives

$$\partial_i U_L = \mathscr{R}_L[\partial_i u] - \frac{\sigma_\alpha}{2}\partial_i \delta_{\mathscr{W}}.$$

Since $\partial_i u = -2x_i/r^4$, this is .

Two derivatives

Two integrations by parts give

$$\alpha \int_{r>\varepsilon} u\,\partial_i\partial_j f \,\mathrm{d}^2x - \alpha \int_{r>\varepsilon} \partial_i\partial_j u\, f \,\mathrm{d}^2x$$

$$= -\alpha\left\{ \varepsilon^{-1}\int \omega_i \partial_j f(\varepsilon\omega)\,\mathrm{d}\varphi + 2\varepsilon^{-2}\int \omega_i\omega_j f(\varepsilon\omega)\,\mathrm{d}\varphi \right\}.$$

Taylor expansion gives the boundary expression

$$-\sigma_\alpha \varepsilon^{-2}\delta_{ij} f(0) - \frac{3\sigma_\alpha}{4}\partial_i\partial_j f(0) - \frac{\sigma_\alpha}{8}\delta_{ij}\Delta f(0) + O(\varepsilon^2 p_4(f)).$$

The power and logarithmic terms are exactly those in the direct finite part of $\partial_i\partial_j u$. Hence

$$\partial_i\partial_j U_L = \mathscr{R}_L[\partial_i\partial_j u] - \frac{3\sigma_\alpha}{4}\partial_i\partial_j\delta_{\mathscr{W}} - \frac{\sigma_\alpha}{8}\delta_{ij}\Delta\delta_{\mathscr{W}}.$$

Taking a trace yields

$$\Delta U_L = 4V_L^c - \sigma_\alpha\Delta\delta_{\mathscr{W}}.$$

Subtracting this trace from the preceding tensor identity and dividing by two gives

$$\frac{1}{2}K_{ij}U_L = S^c_{L,ij} + \sigma_\alpha\Big(\frac{7}{16}\delta_{ij}\Delta - \frac{3}{8}\partial_i\partial_j\Big)\delta_{\mathscr{W}}.$$

This verifies every coefficient in Proposition without numerical execution.

The unconstrained supported extension space

The finite contact dimension should not obscure the larger extension problem.

A translation-invariant scalar difference supported on $\mathscr{W}$ and with transverse scaling degree at most 2 is $c\delta_{\mathscr{W}}$. Without translation invariance its coefficient can be an arbitrary distribution in t, z.

For stress, scaling degree at most 4 permits transverse delta derivatives through order 2. The almost-homogeneity condition reduces these to order 2, and the symmetry and Ward equations then give Theorem . If almost homogeneity is dropped but the other stated symmetries and conservation are kept, one can additionally add

$$T^A{}_B = \lambda L^{-2}\delta^A_B\delta_{\mathscr{W}}, \qquad T^i{}_j = 0.$$

This is conserved for the reference representative, but it violates the declared massless almost-homogeneous class and, unless accompanied by a changed trace prescription, its trace condition.

For a current, a normal-order bound alone likewise does not select circulation or remove possible longitudinal structures if longitudinal Lorentz covariance is dropped. A statement that there are ``only three apex counterterms'' without the assumptions of Theorem would therefore be false.

These observations are elementary instances of the distinction between a finite normal-jet module and a finite-dimensional space of normalization parameters [2, 15]. They also explain why no counterterm class from a different companion observable is identified with the class constructed here.

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