I-B-VP-2: The exact global wavefront relation of the conical Friedrichs ground state

Geometric propagation, apex diffraction, and positive frequency on the prescribed regular Enter spacetime

Ngo Cheung

October 2026

Abstract

We determine the ordinary global wavefront set of the massless holonomy-twisted Friedrichs spectral ground-state covariance on the regular conical spacetime prescribed in I-B. The result is an equality, valid for every opening $0 < \alpha < 1$ and fractional holonomy $0 \le \beta < 1$, including the angular transition where a geometric front meets the apex-diffracted front.

Let $\mathscr{G}^{+}$ be the closure of the positive-frequency relation of regular null geodesics, including its diagonal limit, and let $\mathscr{D}^{+}_{\mathrm{ap}}$ be the positive conormal relation of

$$(t - t')^2 = (r + r')^2 + (z - z')^2.$$

Then

$$\mathrm{WF}(W_\beta) = \begin{cases} \mathscr{G}^{+}, & \alpha = 1/N,\ \beta = p/N, \\ \mathscr{G}^{+} \cup \mathscr{D}^{+}_{\mathrm{ap}}, & \text{otherwise,} \end{cases}$$

where $N \ge 2$ and $p = 0, \dots, N-1$. Thus the full strictly diffractive relation disappears exactly in the finite-character quotient sectors. In other sectors it remains in the joint wavefront set even at isolated angular zeros of its leading coefficient.

The proof sums the Euclidean angular modes before continuation, uses a two-variable meromorphic expression to

control every angular transition, and combines its reduced-variable structure with the bisolution equations. It also gives the explicit inverse-square-root diffractive coefficient and its half-order regularity improvement away from transition angles.

The contribution is a convention-complete global analysis of the specified covariance within the companion framework, not a claim of priority for conical diffraction. It distinguishes local Hadamard subtraction from a global geodesic-only spectrum condition. In particular, supported Green-operator agreement does not identify vacuum covariances, and the charged commutator is not the pointwise imaginary part of one complex bundle kernel. The result resolves V2 for this state and extension at the level of the ordinary two-point wavefront set. It neither restores global hyperbolicity nor constructs an apex field algebra, an interacting product, a measurement acquisition of wavefront data, or a gravitational source law.

Bibliographic record

Title	I-B-VP-2: The exact global wavefront relation of the conical Friedrichs ground state
Author	Cheung, Ngo
Publication date	2026/10/01
Institution	Enter
Report number	I-B-VP-2-v2
Publisher	Cheung Ngo Medical Limited
First page	1
Last page	40
Language	en

Book edition note (t425u)

Standing Enter refusals, scope vocabulary (proved here/supplied input/conditional/ not constructed/not claimed), and Scholar bibliography policy are frozen in the companion chapter *I-B-VP Book Ledger* v1. O.1 remains

NOT_CLOSED; XVI-N is not discharged; XV-B is not opened; no Host C, Page/island, or spacetime-emergence claim; wired=false; SOURCE_DATE_EPOCH=1792627200. This sheet is a companion manuscript record and is *not* claimed to be Google Scholar-indexed. The published Enter Calculus article is cited only for finite-instrument / operational interface discussions, never as a PDE premise.

Book expansion: wavefront notation panel

toprule Item Convention

midrule Covector orientation Fixed before the angular lift

Angular lift Declared with the mode kernel

Geometric relation Distinct from the diffracted relation

Diffracted relation Centred on the *total summed kernel* at transition angles

Wavefront on $M \times M$ Ordinary WF statements; pullbacks/products proved separately

bottomrule

Purpose, background, and corrections to the proposed route

Cheung's I-B prescribes a regular conical spacetime together with a separate finite operational interface [14]. I-B-G constructs intrinsic scalar propagation after a spatial self-adjoint extension has been supplied [15]. I-B-VP-v2 constructs the distinguished massless Friedrichs spectral ground state and establishes its local Hadamard form [16]. I-B-VP-0 evaluates its charged current [17]; I-B-VP-1 constructs prescribed extensions of selected one-point distributions across the apex in an additional smooth completion [18].

The present object is different: the global covariance on the *regular* spacetime M, with both endpoints satisfying $r, r' > 0$. No smooth structure at the apex and no apex-supported one-point counterterm enters the proof.

Established analytical background. Sommerfeld-type cone kernels and cosmic-string Euclidean Green functions have a substantial history [1, 2]. Geometric and diffractive propagation on cones and edges are treated

by Cheeger–Taylor, Melrose–Wunsch, and Melrose–Vasy–Wunsch [3, 4, 5, 6]. The magnetic Aharonov–Bohm diffraction problem provides another relevant model [7]. We use these works for context, not as substitutes for checking the exact holonomy, state, and transition formulas below.

The global statement is proved directly from an explicitly summed kernel. The standard tools needed are distributional boundary values, the wavefront pullback and product theorems, microlocal elliptic regularity, and propagation of singularities for a smooth real-principal-type operator on the regular host [9, 10].

Four necessary corrections

The proposed V2 route requires the following corrections.

1. A retarded transfer theorem is not a state-transfer theorem. Equality of supported retarded or advanced operators in a laboratory window does not imply equality of positive-frequency covariances. The appropriate comparison is equality of the local singular structure, or smoothness of a specified covariance difference, not equality of states.

2. The global relation must include limiting geometric rays. A regular null geodesic can approach a ray through the removed apex. Its limiting covector pair belongs to a closed wavefront set, even though the limiting ray is not a geodesic in M. We therefore use the closure $\mathscr{G}^{+}$, rather than only the relation of actual regular null geodesics.

3. A diffraction bound must allow exact disappearance. Some cone–character pairs admit a finite image formula with no strictly diffractive relation. Conversely, a zero of one leading coefficient at one angular value does not remove a covector from the *joint* wavefront set if nearby angular values retain that covector in the limit.

4. The charged commutator is not obtained by taking a pointwise imaginary part. The spatial bundle kernel can be complex before any Lorentzian time prescription is introduced. The correct commutator uses opposite operator ordering and the conjugate charge sector. The operator-valued time-odd part, not the pointwise imaginary part of W_{β}, is fixed by the causal propagator.

These are corrections to the logical route, not changes to the prescribed geometry, Friedrichs extension, or spectral

state.

Enter and the non-extraction boundary

The Philosophy of Enter distinguishes destructive readout from quantum recovery, a classical record from a decoder, and an analogy from an identification [13]. I-B and En09-A implement a finite destructive instrument returning a classical code and a surviving residual, not the measured quantum clock register [14, 25]. That interface does not select the continuum field, apex extension, vacuum, or wavefront relation.

The background, worldsheet defect, extension, and state remain inputs. A finite conditional density matrix is not the bidistribution analyzed here. A wavefront condition is not an Einstein equation or a derivation of the metric [14, prop:not-source] [19, prop:noeinstein]. The philosophical discussion fixes interpretation; it is not a premise of the continuum proof.

Scope of the new result. We prove an exact ordinary C^{∞}-wavefront theorem on $M \times M$, not a theorem about an unspecified boundary wavefront calculus. We do not claim a complete symbol calculus at the geometric–diffractive transition, a classification of all ground states, or an interacting microlocal algebra. The exact wavefront equality does not require any of those stronger claims.

The regular host and the spectral covariance

Geometry, bundle, and non-global-hyperbolicity

Fix

$$0 < \alpha < 1, \qquad 0 \le \beta < 1.$$

The spacetime and metric are

$$M = \mathbb{R}_t \times (0,\infty)_r \times (\mathbb{R}/2\pi\mathbb{Z})_\varphi \times \mathbb{R}_z, \qquad g = -\mathrm{d}t^2 + \mathrm{d}r^2 + \alpha^2 r^2 \mathrm{d}\varphi^2 + \mathrm{d}z^2.$$

The future orientation is that of ∂_t. The completion adds the worldsheet

$$\mathscr{W} = \mathbb{R}_t \times \{o\} \times \mathbb{R}_z.$$

It is not part of M.

A lifted parallel representation of the charged line bundle is fixed by

$$\Phi(\varphi + 2\pi) = e^{2\pi i\beta}\Phi(\varphi).$$

Changing a local parallel frame multiplies the covariance by a smooth nowhere-zero phase, and therefore does not change its wavefront set.

Write

$$q_N = \mathrm{d}r^2 + \alpha^2 r^2 \mathrm{d}\varphi^2 + \mathrm{d}z^2, \qquad \mathrm{d}V = \alpha r\, \mathrm{d}r\, \mathrm{d}\varphi\, \mathrm{d}z.$$

Let B_β be the nonnegative Friedrichs extension of $-\Delta_{q_N}$ on the twisted spatial Hilbert space. The extension is an input, as in I-B-G [15, 12]. No distributional curvature-coupling interaction at $\mathscr{W}$ is introduced.

Proposition (The regular host is not globally hyperbolic). *The function t is temporal, but M is not globally hyperbolic.*

Proof. The inverse metric gives $g^{-1}(\mathrm{d}t, \mathrm{d}t) = -1$. Choose $R > 0$, $T_0 > R$, and fixed φ_0, z_0. The points

$$p = (-T_0, R, \varphi_0, z_0), \qquad q = (T_0, R, \varphi_0, z_0)$$

have a causal diamond containing

$$x_m = (0, R/m, \varphi_0, z_0), \qquad m \geq 2 :$$

the radial spatial travel length from either endpoint is less than $R < T_0$. This sequence has no convergent subsequence in M. Thus the diamond is not compact.

This reproduces the relevant geometric obstruction of I-B-G prop:nonGH without importing a global Cauchy theorem. A self-adjoint extension supplies the evolution used here; it does not make the regular slices Cauchy surfaces for all causal curves.

Definition and compact-test finiteness

Put $H_\beta = \sqrt{B_\beta}$. The covariance is

$$W_\beta(T; x, x') = \left[\frac{e^{-\mathrm{i}TH_\beta}}{2H_\beta}\right](x, x'), \qquad T = t - t'.$$

For compact spacetime sections,

$$\mathcal{K}_\beta f = \int_{\mathbb{R}} e^{\mathrm{i}tH_\beta}(2H_\beta)^{-1/2} f(t)\,\mathrm{d}t, \qquad W_\beta(f, g) = \langle \mathcal{K}_\beta f, \mathcal{K}_\beta g\rangle.$$

For a real neutral scalar take $\beta = 0$. For a charged scalar use the particle and antiparticle sectors on the bundle and its conjugate, as in I-B-VP-v2 def:state [16].

For completeness, the estimate needed at zero energy is recalled.

Lemma (Infrared control). *Every compact smooth spatial section belongs continuously to* $\mathrm{dom}\,(B_\beta^{-1/4})$ *in its fixed-support test topology. Consequently defines a continuous positive covariance on compact spacetime tests.*

Proof. With

$$\nu_n = \frac{|n + \beta|}{\alpha},$$

the heat kernel is

$$K_\beta(s; x, x') = \frac{e^{-(z - z')^2/(4s)}}{\sqrt{4\pi s}} \frac{e^{-(r^2 + r'^2)/(4s)}}{4\pi\alpha s} \times \sum_{n \in \mathbb{Z}} e^{\mathrm{i}(n + \beta)(\varphi - \varphi')} I_{\nu_n}\left(\frac{rr'}{2s}\right).$$

It follows from the Friedrichs Bessel transform and Weber's integral [11, 15]. On a compact regular spatial set and for $s \geq 1$, the Bessel power series gives

$$|K_\beta(s; x, x')| \leq C s^{-3/2 - \nu_*}, \qquad \nu_* = \frac{\min(\beta, 1 - \beta)}{\alpha}.$$

For a compact spatial section v,

$$\mathrm{lVertB_}\beta^{-1/4}v\|^2 = \frac{1}{\sqrt{\pi}}\int_0^\infty s^{-1/2}\langle v, e^{-sB_\beta}v\rangle\,\mathrm{d}s.$$

Use the contraction bound for $0 < s < 1$, and the displayed compact heat bound with $\mathrm{lVertv}\|_{L^1}^2$ for $s \geq 1$. The latter integrand is $O(s^{-2-\nu_*})$. The bounds are continuous in fixed-support test seminorms. Time integration then defines $\mathscr{K}_\beta$, and its Gram form proves positivity.

The spectral construction is quasifree and has nonnegative one-particle Hamiltonian H_β. It is the specified distinguished spectral state, not a uniqueness theorem among all algebraic ground states.

On regular charts,

$$P_X W_\beta = P_{X'} W_\beta = 0, \qquad P = \partial_t^2 - \Delta_{q_N} = -\mathrm{Box}_g.$$

These equations follow by spectral calculus and integration by parts on compact tests. Their principal symbol is

$$p(X,\kappa) = -\kappa_t^2 + \kappa_r^2 + \frac{\kappa_\varphi^2}{\alpha^2 r^2} + \kappa_z^2,$$

up to an irrelevant overall sign.

Wavefront and frequency conventions

All wavefront sets below are ordinary, unprimed wavefront sets:

$$(X,\kappa;X',\kappa') \in \mathrm{WF}\,(W_\beta)$$

means the covector of the bidistribution on $M \times M$. A conventional canonical-relation notation would replace κ' by $-\kappa'$.

We use the local Fourier convention

$$\hat{u}(\xi) = \int e^{-\mathrm{i}x\cdot\xi}u(x)\,\mathrm{d}x.$$

The factor $e^{-\mathrm{i}\omega(t-t')}$, $\omega > 0$, therefore has first temporal covector $\kappa_t = -\omega < 0$. For signature $(-+++)$, such a null covector is future directed after raising its index. This is our positive-frequency orientation.

An exact summed kernel and its boundary values

Use the differences

$$T = t - t', \qquad Z = z - z', \qquad \delta = \varphi - \varphi'$$

in a chosen angular lift. For $\varepsilon > 0$, define

$$Q_\varepsilon = \frac{r^2 + r'^2 + Z^2 - (T - i\varepsilon)^2}{2rr'}, \qquad \eta_\varepsilon = \operatorname{arcosh} Q_\varepsilon,$$

where the branch has $\operatorname{Re}\eta_\varepsilon > 0$. For real $T \neq 0$, Q_ε approaches the real axis from the side $\operatorname{sgn} T$.

Summation before continuation

Define

$$S_\beta(\eta, \delta) = \sum_{n \in \mathbb{Z}} e^{-|n+\beta|\eta/\alpha} e^{i(n+\beta)\delta}, \qquad \operatorname{Re}\eta > 0.$$

Lemma (Closed angular sum). *For $a = \eta/\alpha$,*

$$S_\beta(\eta, \delta) = \frac{e^{-\beta a + i\beta\delta}}{1 - e^{-a + i\delta}} + \frac{e^{-(1-\beta)a - i(1-\beta)\delta}}{1 - e^{-a - i\delta}}$$

$$= \frac{e^{i\beta\delta}\sinh((1-\beta)a) + e^{-i(1-\beta)\delta}\sinh(\beta a)}{\cosh a - \cos\delta}.$$

The formulas include $\beta = 0$. In particular, the meromorphic continuation of S_β is odd in η.

Proof. Separate $n \geq 0$ and $n \leq -1$ and sum the two convergent geometric series. Putting their terms over a common denominator gives . Its numerator is odd in a and its denominator is even.

Proposition (Global regularized covariance). *For $\varepsilon > 0$,*

$$W_{\beta,\varepsilon}(X, X') = \frac{S_\beta(\eta_\varepsilon, \delta)}{8\pi^2 \alpha rr' \sinh\eta_\varepsilon} = \left[\frac{e^{-(\varepsilon + iT)H_\beta}}{2H_\beta}\right](x, x').$$

*Its distributional boundary value as ε*downarrow$\,0$ *is W_β.*

Proof. For a real Euclidean separation $a_0 > 0$, the spectral identity

$$\frac{e^{-a_0\sqrt{B}}}{2\sqrt{B}} = \int_0^\infty (4\pi s)^{-1/2} e^{-a_0^2/(4s)} e^{-sB}\,\mathrm{d}s$$

and give

$$G_{E,\beta} = \frac{S_\beta(\eta,\delta)}{8\pi^2 \alpha r r' \sinh\eta}, \qquad \cosh\eta = \frac{r^2 + r'^2 + Z^2 + a_0^2}{2rr'}.$$

The heat integral is justified at large s by Lemma . The change of variable $\zeta = rr'/(2s)$ uses

$$\int_0^\infty e^{-\zeta\cosh\eta} I_\nu(\zeta)\,\mathrm{d}\zeta = \frac{e^{-\nu\eta}}{\sinh\eta}$$

[11]. This is also the Euclidean mode kernel of I-B-VP and I-B-VP-0 [16, 17].

Both sides continue holomorphically in a_0 to $\mathrm{Re}\,a_0 > 0$. The denominators in are nonzero there under , since $\mathrm{Re}\,\eta > 0$. Set $a_0 = \varepsilon + \mathrm{i}T$. Spectral dominated convergence on the compact-test form of Lemma proves the boundary assertion.

A boundary-value fact used below

Lemma (One complex variable with smooth parameters). *Let $F(z,y)$ be holomorphic for $\mathrm{Im}\,z > 0$, smooth in real parameters y, and locally bounded by a power of $(\mathrm{Im}\,z)^{-1}$, together with each fixed finite number of y -derivatives. Then it has a distributional boundary value on compact real coordinate patches, and*

$$\mathrm{WF}\,(F(s + \mathrm{i}0, y)) \subset \{(s,y;\lambda,\nu) : \lambda \geq 0\}.$$

For a lower-half-plane boundary value, $\lambda \leq 0$.

The assertion is local and remains valid after a real-analytic change of the complex transverse coordinate with the corresponding half-plane orientation.

Proof. This is the distributional boundary-value theorem with parameters [9]. We recall the two ingredients needed here. Sufficiently many holomorphic primitives in z have controlled boundary limits; differentiating those limits defines the original distributional boundary value. The estimates with parameters make this construction continuous on joint tests.

For the wavefront assertion, localize and Fourier transform. In a cone where $\lambda < -c|(\lambda, v)|$, shift the z contour upward. An almost-analytic extension of the real cutoff makes its $\bar{\partial}$-error vanish to arbitrarily high order at the boundary. That order can be chosen larger than the fixed polynomial growth order of F. The shifted term decays exponentially in the negative λ-cone; the cutoff errors decay to every inverse power. This proves . Reflection proves the lower-half-plane case. The same contour argument in an analytic transverse coordinate proves the last assertion.

For the function

$$\mathscr{T}_\beta(Q, \delta) = \frac{S_\beta(\operatorname{arcosh} Q, \delta)}{\sinh(\operatorname{arcosh} Q)},$$

the required polynomial bounds hold near $Q = -1$, uniformly on compact real angular intervals. Indeed, $\operatorname{Re} \operatorname{arcosh} Q \geq c \operatorname{Im} Q$ on a sufficiently small upper-half-plane compact patch, the geometric denominators in are bounded below by a constant times $\operatorname{Re} \operatorname{arcosh} Q$, and $\sinh(\operatorname{arcosh} Q)$ has only a square-root zero. Angular differentiation changes the polynomial bound, but not its existence.

Consequently, near a point with $T \neq 0$, the covariance has the reduced form

$$W_\beta = \frac{1}{8\pi^2 \alpha r r'} \mathscr{T}_\beta(Q + \mathrm{i}0 \operatorname{sgn} T, \delta), \qquad Q = \frac{r^2 + r'^2 + Z^2 - T^2}{2rr'}.$$

The analytic change from T to Q is regular there, since $\partial_T Q = -T/(rr') \neq 0$. This observation is particularly useful at an angular transition: no moving image indicator is differentiated.

The local direct term

Lemma (Local Hadamard form from the closed expression). *On a sufficiently small developing neighbourhood of each regular point, in a local parallel frame,*

$$W_\beta(X, X') = \frac{1}{4\pi^2(|\mathbf{X} - \mathbf{X}'|^2 - (T - \mathrm{i}0)^2)} + H_\beta(X, X'),$$

with H_β smooth; in fact the local regular remainder is real-analytic in the corresponding real variables.

Proof. Take a lift with δ near zero and Q near 1. The quotient in is meromorphic in (Q, δ), because both its numerator divided by $\sinh \eta$ and its denominator are even analytic functions of η near $\eta = 0$. Its denominator factors as

$$\cosh(\operatorname{arcosh} Q/\alpha) - \cos\delta = (Q - \cos(\alpha\delta))h(Q, \delta), \qquad h(1, 0) = \alpha^{-2}.$$

For $\delta \neq 0$ sufficiently small, either term in computes the pole residue. It gives

$$\mathscr{F}_{\beta}(Q, \delta) - \frac{\alpha}{Q - \cos(\alpha\delta)}$$

analytic across the pole. Analytic division extends the same statement through $(Q, \delta) = (1, 0)$. Multiplication by the prefactor in gives the direct kernel in , because

$$2rr'(Q - \cos(\alpha\delta)) = r^2 + r'\text{^}2 - 2rr'\cos(\alpha\delta) + Z^2 - T^2.$$

The remaining analytic function has a smooth Lorentzian boundary value. The direct term is the ordinary Minkowski positive-frequency distribution in the developing chart.

This independently recovers the local statement used in I-B-VP and I-B-VP-0. The global proof below is not an application of a globally hyperbolic local-to-global theorem.

The geometric and apex relations

For a chosen lift define

$$\theta_k = \alpha(\delta - 2\pi k), \qquad k \in \mathbb{Z},$$

$$R_k^2 = r^2 + r'\text{^}2 - 2rr'\cos\theta_k + Z^2, \qquad Q_k = R_k^2 - T^2.$$

Only finitely many k satisfy $|\theta_k| \leq \pi$ on a fixed compact lifted angular patch.

Definition (Closed geometric positive relation). Let $\mathscr{G}^{+}_{\mathrm{reg}}$ be the ordinary positive-frequency covector relation of null geodesic segments entirely in M. It includes

$$(X, \kappa; X', -\kappa')$$

when κ' is parallel transported to κ along that segment and κ is future directed. Define

$$\mathscr{G}^{+} = \overline{\mathscr{G}^{+}_{\mathrm{reg}}} \subset T^{*}(M \times M) \setminus 0.$$

The closure includes the standard diagonal null relation.

Away from the diagonal, the actual regular geodesic relation is locally described by

$$Q_k = 0, \qquad |\theta_k| < \pi, \qquad (\kappa, \kappa') = \lambda \, \mathrm{d}_{X,X'} Q_k, \qquad \lambda T > 0.$$

Its closure allows $|\theta_k| = \pi$.

Lemma (Geometric interpretation). *Formula describes all regular null geodesic segments. The additional off-diagonal limits in $\mathscr{G}^{+}$ occur at $|\theta_k| = \pi$.*

Proof. A geodesic away from the apex develops into a straight line in the transverse plane, together with an affine longitudinal coordinate. A straight transverse segment avoiding the origin has a continuous angular change strictly between $-\pi$ and π. Its length is R_k for the corresponding lift. The null condition is $|T| = R_k$. Differentiating its squared separation gives the parallel endpoint covectors in .

A limit with angular change π or $-\pi$ passes through the transverse origin. Conversely, those segments are limits of straight segments avoiding the origin. The local finiteness of the allowable lifts excludes other off-diagonal limits.

Put

$$L_{\mathrm{ap}} = \sqrt{(r + r')^2 + Z^2}, \qquad F_{\mathrm{ap}} = (r + r')^2 + Z^2 - T^2.$$

Definition (Apex-diffracted positive relation). Define

$$\mathscr{D}^{+}_{\mathrm{ap}} = \left\{ (X, \kappa; X', \kappa') : \begin{array}{c} lF_{\mathrm{ap}} = 0, \\ (\kappa, \kappa') = \lambda \, \mathrm{d}_{X,X'} F_{\mathrm{ap}}, \\ \lambda T > 0 \end{array} \right\}.$$

No angular covector occurs in this relation. The angular transition set is

$$\mathscr{B}_{\alpha} = \{\delta \bmod 2\pi : \cos\delta = \cos(\pi/\alpha)\}.$$

Both endpoint covectors in are nonzero and null. For example,

$$\mathrm{d}_X F_{\mathrm{ap}} = -2T\,\mathrm{d}t + 2(r + r')\,\mathrm{d}r + 2Z\,\mathrm{d}z.$$

The first raised covector is future directed precisely when $\lambda T > 0$.

Proposition (The broken ray and longitudinal matching). *For $T = L_{\mathrm{ap}} > 0$, the relation $\mathscr{D}^{+}_{\mathrm{ap}}$ is the endpoint relation of an incoming radial null segment and an outgoing radial null segment meeting the removed worldsheet at*

$$z_* = \frac{rz' + r'z}{r + r'}, \qquad t_* = t' + \frac{r'}{r + r'}T.$$

The energy and longitudinal momentum agree across that meeting. The incoming and outgoing angular positions need not agree. Moreover,

$$\mathscr{D}^{+}_{\mathrm{ap}} \cap \mathscr{G}^{+}$$

consists exactly of these radial covector pairs at the angular transition $\delta \in \mathscr{B}_{\alpha}$.

Proof. Minimize the length of a two-segment spatial path through the worldsheet:

$$\sqrt{r'^2 + (z_* - z')^2} + \sqrt{r^2 + (z - z_*)^2}.$$

Equality of the two longitudinal slopes gives and total length L_{ap}. The corresponding null segments have common longitudinal velocity Z/T and transverse speed $(r + r')/T$. The covectors in record the same energy and longitudinal momentum.

For a limiting geometric ray to have this length,

$$R_k^2 = L_{\mathrm{ap}}^2 \quad \Leftrightarrow \quad \cos\theta_k = -1.$$

With $|\theta_k| \le \pi$, this means $\theta_k = \pi$ or $-\pi$, equivalently $\delta \in \mathscr{B}_{\alpha}$. At that point $\mathrm{d}Q_k = \mathrm{d}F_{\mathrm{ap}}$.

The meeting point is not an added field-theoretic degree of freedom. It is a description of the endpoint relation selected by the Friedrichs kernel.

Global wavefront containment, including every transition

Proposition (Global positive-frequency upper bound). *For every* $0 < \alpha < 1$ *and* $0 \le \beta < 1$,

$$\mathrm{WF}\,(W_\beta) \subset \mathscr{G}^+ \cup \mathscr{D}^+_{\mathrm{ap}}.$$

In particular, there are no one-zero-endpoint covectors and no additional pure angular wavefronts at an image boundary.

Proof. The proof separates the real values of Q in .

1. The regions Q > 1 and Q < −1. For $Q > 1$, $\eta = \operatorname{arcosh} Q$ is real and positive, and the summed expression is smooth. For $Q < -1$, its two boundary values have

$$\eta = \nu + \mathrm{i}\pi \operatorname{sgn} T, \qquad \nu > 0.$$

The geometric denominators in are nonzero because $e^{-\nu/\alpha} < 1$. The factor $\sinh \eta$ is also nonzero. Thus each boundary value is smooth.

2. The interval −1 < Q < 1. Write

$$\eta = \mathrm{i}\gamma \operatorname{sgn} T, \qquad 0 < \gamma < \pi.$$

The only poles in occur when

$$\gamma = |\alpha(\delta - 2\pi k)| = |\theta_k|, \qquad |\theta_k| < \pi.$$

They are precisely $Q_k = 0$. Each of the two geometric fractions has a simple pole in a smooth defining function for the corresponding front. Even where two fronts cross in the base, the two fractions provide the separate conormal descriptions. Consequently the wavefront is contained in their conormals.

The half-plane boundary-value orientation in Lemma gives a nonnegative coefficient of $\mathrm{d}Q$ for $T > 0$, and a

nonpositive one for $T < 0$. Since

$$\partial_t Q = -\frac{T}{rr'},$$

the first temporal covector is negative. This is exactly the orientation $\lambda T > 0$ in .

3. The branch point Q = 1. At an angular value not congruent to zero, the evenness in η makes analytic in Q. Near $\delta = 0 \bmod 2\pi$, the factorization in the proof of Lemma gives the direct geometric conormal. At a diagonal point the complete local wavefront is the usual positive-frequency diagonal null relation by .

4. The branch point Q = −1, including δ ∈ $\mathscr{B}$_{α}. Here $T \neq 0$. The map

$$(X, X') \mathrm{longmapsto} (Q, \delta)$$

is a submersion, and expresses the kernel as the pullback of a two-variable distribution, multiplied by the smooth factor $1/(8\pi^2 \alpha r r')$. The pullback theorem therefore gives

$$(\kappa, \kappa') = \lambda \, \mathrm{d}Q + \nu \, \mathrm{d}\delta$$

for every possible wavefront covector at this branch point. There is no contribution from passive coordinates.

On $F_{\mathrm{ap}} = 0$,

$$\mathrm{d}Q = \frac{1}{2rr'} \mathrm{d}F_{\mathrm{ap}}.$$

The first endpoint part of dQ is null and has no angular component. Hence

$$p(X, \kappa) = \frac{\nu^2}{\alpha^2 r^2}.$$

The bisolution equation and microlocal elliptic regularity force $p(X, \kappa) = 0$, so $\nu = 0$. The total covector is nonzero, therefore $\lambda \neq 0$; both endpoint covectors are nonzero. Lemma fixes its positive-frequency orientation. This is precisely $\mathscr{D}^{+}_{\mathrm{ap}}$.

This last argument applies to the *total summed kernel*. It does not assign or differentiate separate values of moving

image-sector indicators.

5. Exhaustion. At $T = 0$, the only possible singular points are on the diagonal: indeed $Q \geq 1$, with equality only when $r = r'$ and $Z = 0$, and a singular angular value then requires $\delta = 0 \bmod 2\pi$. The preceding cases cover all of $M \times M$.

Remark (Why support or a remainder estimate would not suffice). The proof uses a reduced-variable pullback theorem, an oriented boundary-value theorem, and the characteristic equations. Neither finite propagation nor a finite-order estimate alone would prove . This is the distinction between distributional control and a wavefront theorem also emphasized in the supplied Em01-D material [23].

The geometric lower bound

Proposition (Every closed geometric covector occurs). *For all parameters,*

$$\mathscr{G}^{+} \subset \mathrm{WF}\,(W_{\beta}).$$

Proof. Lemma gives every positive diagonal null covector. Fix a regular null geodesic segment starting at X'. Propagation of singularities for P_X, with the primed variables retained as parameters, transports the first covector along that entire segment [10]. The path remains in the smooth regular host; no passage through the apex is made in this argument. Thus every element of $\mathscr{G}^{+}_{\mathrm{reg}}$ occurs. The wavefront set is closed in the punctured cotangent bundle, so it contains $\mathscr{G}^{+}$.

One can also read the nonzero coefficient directly from : a geometric branch has principal term

$$\frac{e^{2\pi i\beta k}}{4\pi^{2}(R_{k}^{2} - (T - \mathrm{i}0)^{2})}.$$

Its character factor is never zero. Propagation avoids any need to select a leading term at a crossing of geometric fronts.

The diffracted coefficient and its nonvanishing criterion

Set

$$c = \frac{\pi}{\alpha}.$$

On the complement of the angular transition set, define

$$\mathscr{A}_{\alpha,\beta}(\delta) = \frac{e^{i\beta\delta}\sin((1-\beta)c) + e^{-i(1-\beta)\delta}\sin(\beta c)}{\cos c - \cos\delta}.$$

This is a complex, real-analytic angular function in each transition-free angular interval.

Lemma (Exact local form at a strictly diffracted front). *Near a point with $F_{\mathrm{ap}} = 0$ and $\delta \notin \mathscr{B}_{\alpha}$, there are smooth functions A_* and B_*, analytic in the local real variables, such that*

$$W_{\beta} = A_*(F_{\mathrm{ap}} + i0\,\mathrm{sgn}\,T)^{-1/2} + B_*,$$

and

$$A_*|_{F_{\mathrm{ap}}=0} = \frac{\mathscr{A}_{\alpha,\beta}(\delta)}{8\pi^2\alpha\sqrt{rr'}}.$$

The square root is the principal boundary value.

Proof. Take first $T > 0$. Near $Q = -1$, write $\eta = i\pi + w$. Since

$$Q = -\cosh w, \qquad Q + 1 = \frac{F_{\mathrm{ap}}}{2rr'},$$

one has

$$w = -i\sqrt{\frac{F_{\mathrm{ap}} + i0}{rr'}}(1 + O(F_{\mathrm{ap}})),$$

$$\sinh\eta = i\sqrt{\frac{F_{\mathrm{ap}} + i0}{rr'}}(1 + O(F_{\mathrm{ap}})).$$

The angular denominator in is nonzero at $w = 0$, and

$$S_\beta(\mathrm{i}\pi, \delta) = \mathrm{i}\,\mathscr{A}_{\alpha,\beta}(\delta).$$

Expand $S_\beta(\mathrm{i}\pi + w, \delta)/\sinh(\mathrm{i}\pi + w)$ in its convergent Laurent series in w. Its even and odd powers separate into an analytic function of F_{ap} plus $(F_{\mathrm{ap}} + \mathrm{i}0)^{-1/2}$ times another analytic function. Multiplication by the prefactor in gives . For $T < 0$, expand at $\eta = -\mathrm{i}\pi$. Oddness of S_β and of $\sinh\eta$ gives the same coefficient and the lower boundary value.

Lemma (Wavefront of the boundary square root). *At* $s = 0$,

$$\mathrm{WF}\,((s + \mathrm{i}0)^{-1/2}) = \{(0, \lambda) : \lambda > 0\},$$

and the lower boundary value has $\lambda < 0$. *Multiplication by a smooth coefficient nonzero on* $s = 0$ *preserves that conormal wavefront.*

Proof. The representation

$$(s + \mathrm{i}0)^{-1/2} = \frac{e^{-\mathrm{i}\pi/4}}{\sqrt{\pi}} \int_0^\infty e^{\mathrm{i}\lambda s} \lambda^{-1/2}\, \mathrm{d}\lambda$$

is understood as a damped oscillatory integral. It gives the one-sided wavefront and an elliptic conormal amplitude at every positive λ. The lower boundary value is obtained by conjugating the scalar model. Smooth elliptic multiplication preserves the wavefront [9].

Definition (Finite-character quotient parameters). Define

$$\mathscr{Q} = \left\{ (\alpha, \beta) : \alpha = \frac{1}{N}, \quad \beta = \frac{p}{N}, \quad N \in \mathbb{N},\ N \geq 2, \quad 0 \leq p < N \right\}.$$

Proposition (Exact criterion for absence of a strict diffractive coefficient). *The function* $\mathscr{A}_{\alpha,\beta}$ *vanishes identically on a nonempty transition-free angular interval if and only if* $(\alpha, \beta) \in \mathscr{Q}$.

If $(\alpha, \beta) \notin \mathscr{Q}$, *its nonzero angular values are dense in the complement of* $\mathscr{B}_\alpha$, *and*

$$\mathscr{D}^+_{\mathrm{ap}} \subset \mathrm{WF}\,(W_\beta).$$

Proof. Multiplying the numerator in by $e^{-\mathrm{i}\beta\delta}$ gives

$$\sin((1-\beta)c) + e^{-\mathrm{i}\delta}\sin(\beta c).$$

It vanishes on an interval precisely when both constants vanish:

$$\frac{\beta}{\alpha} \in \mathbb{Z}, \qquad \frac{1-\beta}{\alpha} \in \mathbb{Z}.$$

Their sum gives $1/\alpha = N \in \mathbb{N}$, $N \geq 2$, and then $\beta = p/N$ with $0 \leq p < N$. The converse is immediate.

Outside this exceptional case, the analytic numerator has no open zero interval. At every nonzero coefficient value, Lemmas and give all of $\mathscr{D}^{+}_{\mathrm{ap}}$ over that angular value. Closedness of the joint wavefront set supplies the same radial covectors at its angular zeros and at the transition angles. This proves .

Remark (A zero on an angular slice is not joint smoothness). At a special fixed value of δ, a restricted scalar coefficient can vanish. That is not a neighbourhood on $M \times M$. The lower bound above is a joint statement in the angular and remaining variables. No inference from a single angular slice is used.

Comparison with the image–diffraction representation

For orientation and normalization, define

$$\rho(u)^2 = r^2 + r'^2 + 2rr'\cosh u + Z^2,$$

$$\Phi_\beta(u,\delta) = \sum_{n\in\mathbb{Z}} \sin(\pi\nu_n) e^{-\nu_n u} e^{\mathrm{i}(n+\beta)\delta}, \qquad u > 0.$$

The notation here uses δ, rather than the developing angle $\alpha\delta$, as its second argument.

The Bessel angular identity gives, off the transition angles,

$$W_\beta = \sum_{\text{substack}\, k\in\mathbb{Z}} -\frac{1}{4\pi^3\alpha}\int_0^\infty \frac{\Phi_\beta(u,\delta)}{\rho(u)^2 - (T-\mathrm{i}0)^2}\,\mathrm{d}u.$$

One obtains this formula by applying the heat-kernel angular summation before integrating the Euclidean time heat factor, and then continuing its *total* kernel. It is the covariance counterpart of the

Sommerfeld–Carslaw–Dowker formulas in I-B-G and I-B-VP [15, 16]. At transition angles the joint distribution, not separate pointwise sector values, is the object.

The coefficient in can be checked without assigning a value to a divergent angular series:

$$\Phi_\beta(u,\delta) = \frac{S_\beta(u - \mathrm{i}\pi, \delta) - S_\beta(u + \mathrm{i}\pi, \delta)}{2\mathrm{i}}, \qquad u > 0.$$

Its transition-free limit is therefore

$$\Phi_\beta(0,\delta) = -\mathscr{A}_{\alpha,\beta}(\delta).$$

Since

$$\rho(u)^2 - (T - \mathrm{i}0)^2 = F_{\mathrm{ap}} + rr'u^2 + O(u^4) + \mathrm{i}0\,\operatorname{sgn} T,$$

the endpoint model

$$\int_0^\infty \frac{\mathrm{d}u}{s + rr'u^2} = \frac{\pi}{2\sqrt{rr'}}\, s^{-1/2}$$

recovers . The full wavefront proof at the angular transition was instead given by the reduced-variable argument of Proposition ; this endpoint check is not used to differentiate a singular moving image sector.

Corollary (Relative strength away from transitions). *At a nonzero diffractive coefficient and away from geometric fronts, the transverse Fourier amplitude of W_β has order $-1/2$. At a single noncoincident geometric front away from other fronts it has order 0. Thus the strictly diffracted covariance is one-half derivative smoother in its normal variable.*

Equivalently, at such elliptic points the local normal Sobolev thresholds are $s < 0$ for the diffracted front and $s < -1/2$ for the geometric front. No uniform half-order assertion at their transition is made.

Proof. The square-root model has Fourier amplitude $\lambda^{-1/2}$ on a half-line. The simple-pole geometric model has amplitude of order 0. After localization, the corresponding one-dimensional Sobolev integrals have thresholds

$$\int_1^\infty \lambda^{2s-1}\,\mathrm{d}\lambda < \infty \quad \Leftrightarrow \quad s < 0,$$

and

$$\int_1^\infty \lambda^{2s}\,\mathrm{d}\lambda < \infty \quad \Leftrightarrow \quad s < -1/2.$$

The nonzero leading coefficients make these thresholds sharp.

The quotient sectors and the exact global theorem

An independent finite-image construction

Proposition (Finite-character covariance). *If $\alpha = 1/N$ and $\beta = p/N$, then*

$$W_{p/N}(X, X') = \sum_{k=0}^{N-1} e^{2\pi i p k/N} W_{\mathrm{flat}}(\widetilde{X}, \mathscr{R}_k \widetilde{X}'),$$

where

$$\widetilde{X} = (t, r\cos(\varphi/N), r\sin(\varphi/N), z),$$

$\mathscr{R}_k$ rotates the covering transverse plane by $2\pi k/N$, and

$$W_{\mathrm{flat}}(Y, Y') = \frac{1}{4\pi^2(|\mathbf{Y} - \mathbf{Y}'|^2 - (t - t' - \mathrm{i}0)^2)}.$$

Consequently,

$$\mathrm{WF}(W_{p/N}) = \mathscr{G}^+.$$

Proof. Apply the character-weighted image sum first to the flat heat kernel. Its angular Fourier sum projects covering modes $m \in \mathbb{Z}$ to

$$m = Nn + p,$$

because

$$\sum_{k=0}^{N-1} e^{2\pi \mathrm{i}(p-m)k/N} = \begin{cases} N, & m \equiv p \bmod N, \\ 0, & \text{otherwise.} \end{cases}$$

The projected angular factor is $e^{\mathrm{i}(n + p/N)(\varphi - \varphi')}$, and the Bessel order is $|Nn + p| = |n + \beta|/\alpha$. The factor N supplies

precisely the quotient normalization $1/\alpha$ relative to the conical volume. Thus the result is the Friedrichs heat kernel . Euclidean integration and spectral continuation give .

Every image has the ordinary flat null wavefront relation. If its straight segment avoids the origin, it gives a regular geodesic on the quotient. If it crosses the origin, its endpoint covector is a limit of such geodesics. Thus the finite sum has wavefront contained in $\mathscr{G}^+$. Proposition gives the reverse inclusion. This argument uses the full finite image sum, including boundary representatives when N is even.

Exact global wavefront equality

Theorem (Global Friedrichs wavefront relation). *For the massless Friedrichs spectral covariance on the regular host ,*

$$\operatorname{boxed}\mathrm{WF}(W_\beta) = \begin{cases} \mathscr{G}^+, & (\alpha,\beta) \in \mathscr{Q}, \\ \mathscr{G}^+ \cup \mathscr{D}^+_{\mathrm{ap}}, & (\alpha,\beta) \notin \mathscr{Q}. \end{cases}$$

The equality holds on all of $M \times M$, including the diagonal, crossings of geometric fronts, and every geometric–diffractive angular transition.

Every first endpoint covector is future null, every second endpoint covector is past null, and neither endpoint covector is zero.

Proof. Proposition gives the global upper bound. Proposition gives the geometric lower bound. Outside $\mathscr{Q}$, Proposition gives the entire diffracted lower bound. Inside $\mathscr{Q}$, Proposition removes every strictly diffractive covector outside $\mathscr{G}^+$. The orientation and nonzero-endpoint statements were established in Proposition and the definitions of the two relations.

Corollary (Singular support and the smooth tail). *The singular support is the projection to the base of the relation in . In particular:*

1. *off the diagonal, geometric singularities occur at the regular or limiting geometric travel times $T = \pm R_k$;*

2. *in nonquotient sectors, the additional singular front is $T = \pm L_{\mathrm{ap}}$;*

3. *the open region $T^2 > L_{\mathrm{ap}}^2$ is smooth. The third statement does not say that the covariance or causal kernel vanishes there.*

Proof. The first two statements follow from the exact wavefront equality. For the third, $Q < -1$, where the summed boundary expression is smooth by the first step of Proposition .

Remark (What the result contributes). The exact equality, the treatment of all angular transitions, and the holonomy criterion for disappearance of the strictly diffractive relation are the mathematical release of this companion. Cone diffraction and quotient image methods are established subjects [3, 4, 5, 7]; the present paper does not claim that these phenomena originate in the Enter framework.

Positive frequency, the commutator, and the Hadamard distinction

The correct charged ordering identity

Define

$$E_\beta(T) = \frac{\sin(TH_\beta)}{H_\beta}.$$

The bundle parameter $-\beta$ below is understood modulo one.

Proposition (Spectral splitting and charged commutator). *At the operator-kernel level,*

$$\frac{W_\beta(T) + W_\beta(-T)}{2} = \frac{\cos(TH_\beta)}{2H_\beta},$$

$$W_\beta(T) - W_\beta(-T) = -\mathrm{i}E_\beta(T).$$

For the charged spacetime kernels, the second term in is the reversed opposite-charge ordering:

$$W_\beta(X, X') - W_{-\beta}(X', X) = -\mathrm{i}E_\beta(X, X').$$

Thus the state has the extension-selected canonical commutator. The pointwise real and imaginary parts of the scalar expression for W_β in a parallel frame are not, in general, the two operators in –.

Proof. The first two identities are the elementary spectral cosine and sine identities for . Complex conjugation of the spatial bundle character, followed by endpoint reversal, identifies

$$W_{-\beta}(-T; x', x) = W_{\beta}(-T; x, x').$$

This can be checked directly by reindexing the angular modes. The antiparticle Fock construction gives precisely this reversed operator ordering, proving .

In a twisted parallel representation, the angular factors $e^{\mathrm{i}(n+\beta)\delta}$ can make the time-even spatial kernel complex. Therefore scalar complex conjugation at the same ordered pair is not the operation $T \mapsto -T$ with spatial endpoints retained.

For the neutral real scalar, this reduces to

$$[\varphi(X), \varphi(X')] = -\mathrm{i}E_0(X, X').$$

The sign agrees with the spectral convention in the supplied I-B-G and I-B-VP records [15, 16].

Let $\mathscr{G}^- = -\mathscr{G}^+$ and $\mathscr{D}^-_{\mathrm{ap}} = -\mathscr{D}^+_{\mathrm{ap}}$, where the minus sign negates cotangent fibers without changing base points.

Corollary (Wavefront of the causal kernel). *The wavefront of E_β is the union of both frequency orientations of :*

$$\mathrm{WF}(E_\beta) = \begin{cases} \mathscr{G}^+ \cup \mathscr{G}^-, & (\alpha, \beta) \in \mathscr{Q}, \\ \mathscr{G}^+ \cup \mathscr{G}^- \cup \mathscr{D}^+_{\mathrm{ap}} \cup \mathscr{D}^-_{\mathrm{ap}}, & (\alpha, \beta) \notin \mathscr{Q}. \end{cases}$$

Proof. Equation writes E_β as the difference of two kernels with opposite, disjoint temporal orientations. Theorem , applied to time reversal, identifies the two sets. A microlocal singularity in one temporal orientation cannot be cancelled by a kernel smooth in that orientation.

Finite propagation of the Friedrichs evolution gives causal commutator support for the intrinsic spatial length metric, as in I-B-G thm:speed [15]. This support statement is compatible with the diffracted relation; it is not a support statement for W_β itself.

Local Hadamard form is not a geodesic-only global equality

Radzikowski's theorem relates the Hadamard condition and an oriented geodesic wavefront condition under its globally hyperbolic free-field hypotheses [8]. Here one can still *define* the relation of ordinary regular null geodesics, but the hypotheses of that global theorem do not hold.

Proposition (The additional global relations are genuine). *The covariance is locally of Hadamard form at every regular point. Nevertheless:*

1. *outside $\mathscr{Q}$, its global wavefront contains all strictly diffractive pairs in $\mathscr{D}^{+}_{\mathrm{ap}} \setminus \mathscr{G}^{+}$, which are not linked by a regular null geodesic with the displayed endpoint covectors;*

2. *even in a finite-character quotient sector, the closed geometric relation can contain limiting rays through the removed apex that are not actual regular geodesics.*

Proof. Local Hadamard form is Lemma . For (i), choose $\delta \notin \mathscr{B}_{\alpha}$ on the apex front. Proposition shows that its radial covector pair is outside $\mathscr{G}^{+}$, while Theorem places it in $\mathrm{WF}(W_{\beta})$. A regular geodesic carrying that radial covector cannot avoid the apex and connect the two endpoints.

For (ii), the finite image kernel includes a flat null segment whose transverse projection crosses the covering origin whenever the corresponding image endpoints are antipodal. It is the limit of regular quotient geodesics and hence belongs to $\mathscr{G}^{+}$, but its limiting radial segment is not in M.

The exact result is therefore a positive-frequency spectrum condition for a specified geometric-plus-apex relation. Calling it a geodesic-only globally Hadamard theorem would discard part of the actual wavefront. Conversely, its additional global singularities do not invalidate local Hadamard point splitting.

Products, restrictions, and the one-point companions

Same-orientation products

Proposition (Products justified by the global orientation). *Let $W_{\beta_1}, \dots, W_{\beta_m}$ be covariances of the form*

considered here, with the same time orientation. Their product as kernels on $M \times M$, with the appropriate tensor-product bundle, is defined by the ordinary wavefront product theorem. The same holds after any fixed finite differentiations of the factors.

This assertion does not supply a product between arbitrary oppositely oriented kernels or an interacting algebra.

Proof. Every nonzero wavefront covector of every factor has a future-null first component. A finite sum of nonzero future-causal covectors cannot vanish. Thus no selection of wavefront covectors from the factors can sum to the zero covector on $M \times M$. The product criterion applies [9]. Differentiation does not enlarge a wavefront set.

For example, the ordinary sufficient criterion fails for $W_\beta \overline{W_\beta}$ at an elliptic geometric null covector: the two factors have opposite covectors there. That criterion failure is not a universal nonexistence theorem for every renormalized prescription.

The examples in VII-H and the declared polynomial domains in XI-E are methodological comparisons [21, 22]. Their observables and products are not identified with the present conical bundle kernels.

Equal-time and timelike restrictions

Proposition (Two useful pullbacks). *The equal-time restriction of W_β exists and equals the spatial covariance $1/(2H_\beta)$. For any smooth timelike curve γ in M, the pullback*

$$(\gamma \times \gamma)^* W_\beta$$

also exists as a bidistribution on its parameter domain.

Proof. A covector normal to an equal-time embedding has only temporal components at its endpoints. A nonzero null covector cannot be purely temporal, so the normal set is disjoint from $\mathrm{WF}(W_\beta)$. The spectral restriction is therefore legitimate and gives $1/(2H_\beta)$.

A nonzero null covector cannot annihilate a timelike tangent. Hence neither endpoint of a wavefront pair can lie in the annihilator required to obstruct the timelike pullback. The wavefront pullback theorem applies [9].

The diagonal is different

Proposition (Point splitting still requires subtraction). *The unrenormalized diagonal pullback of W_β fails the ordinary wavefront criterion. In contrast, the local remainder H_β in has well-defined diagonal jets of every finite order. Consequently the Wick square, stress tensor, and current prescriptions of the regular-part companions remain well-defined at every point of M.*

Proof. The diagonal wavefront contains

$$(X, \kappa; X, -\kappa), \qquad \kappa \neq 0, \quad \kappa\text{future null.}$$

These covectors are normal to the diagonal embedding. Thus one cannot justify the unrenormalized pullback by the ordinary criterion. The subtracted remainder is smooth in a neighbourhood of each regular diagonal point, so all its required local diagonal derivatives exist.

The apex front cannot meet the spacetime diagonal: at $X = X'$, $F_{\mathrm{ap}} = 4r^2 > 0$. The additional global fronts therefore do not alter this local subtraction argument.

For the charged current, one must retain the complex first jet of H_β, as in I-B-VP-0 [17, rem:imaginary, def:current]. The global theorem does not turn that prescription into the real-part stress prescription.

The apex finite parts of I-B-VP-1 concern one-point distributions in a separately declared completion [18]. Their contact parameters neither change W_β on $M \times M$ nor provide an extension of its two-point kernel to $\mathscr{W}$.

A regular timelike diffraction echo

The global relation has a direct consequence for an already well-defined restriction, without constructing a detector model.

Fix a static worldline

$$\gamma(u) = (u, r_0, \varphi_0, z_0), \qquad r_0 > 0.$$

Its proper-time parameter is u.

Corollary (A nonzero delayed singularity on a static restriction). *Suppose*

$$\cos(\pi/\alpha) \neq 1, \qquad \mathscr{A}_{\alpha,\beta}(0) \neq 0.$$

Then, near $u - u' = \pm 2r_0$,

$$(\gamma \times \gamma)^* W_\beta = \frac{\mathscr{A}_{\alpha,\beta}(0)}{8\pi^2 \alpha r_0}(4r_0^2 - (u - u' - \mathrm{i}0)^2)^{-1/2}$$

+terms of lower singular order and smooth terms.

The restricted covariance therefore has a genuine delayed singularity at each of those nonzero proper-time separations.

For example, $\alpha = 2/3$, $\beta = 0$ *gives* $\mathscr{A}_{\alpha,\beta}(0) = 1$, *and the hypotheses hold.*

Proof. The pullback exists by Proposition . On this worldline, $\delta = Z = 0$ and $F_{\mathrm{ap}} = 4r_0^2 - (u - u')^2$. The first condition in excludes an angular transition at $\delta = 0$. Every geometric travel time at this fixed spatial point is strictly below $2r_0$ unless it is a transition, so no other front meets a sufficiently small neighbourhood of either displayed separation. Lemma and now give . Nonvanishing of the coefficient and Lemma prove the singularity.

For the example, $c = 3\pi/2$, so

$$\mathscr{A}_{\alpha,\beta}(0) = \frac{\sin(3\pi/2)}{\cos(3\pi/2) - 1} = 1.$$

This is a statement about a free-field covariance restricted to a prescribed worldline. It does not assert a detector response rate, an acquisition channel, or the performance of an Enter readout. At angular or holonomy values excluded by , this corollary makes no conclusion from a vanishing leading coefficient.

Supported laboratories, reporting, and non-transfer

What supported Green agreement actually implies

I-B-G's supported-transfer result concerns retarded and advanced scalar operations in a signed arrival window, with independent retained-shadow and compact-reserve conditions [15, thm:transfer, ass:transfer-contract]. The

corresponding ST sheets retain those support obligations [19, 20].

It is not a theorem that $W_\beta = W_{\mathrm{flat}}$ in that window. Already for $\beta = 0$, the regular diagonal difference is

$$[H_0] = \frac{\alpha^{-2} - 1}{48\pi^2 r^2},$$

which is nonzero for $0 < \alpha < 1$ [16, thm:phi2]. Thus exact covariance equality would contradict the regular-part Wick square.

The correct local statement is

$$W_\beta - W_{\mathrm{flat}} \in C^\infty$$

on a sufficiently small developing product neighbourhood, and on any further specified pair region where the direct singularity has been subtracted and no additional front is met. The explicit kernel and Theorem , not a state-transfer inference, justify that assertion.

Finite records do not determine a wavefront

The finite supplied-report interface can install words describing

$$(\alpha, \beta, \text{bundle convention}, \text{formula}, \text{microlocal localization}, \text{status}).$$

For a calibrated clock basis, put

$$D_j = \langle \tau_j|_C \otimes I_A, \qquad \mathscr{M}(X) = \sum_j \mathrm{lvertj}, w_j\rangle\mathrm{langlej}, w_j|_R \otimes D_j X D_j^\dagger.$$

Then

$$\sum_j D_j^\dagger D_j = I_{CA}, \qquad \mathrm{Tr}_R\, \mathscr{M}(X) = \mathrm{Tr}_C X,$$

also with every untouched finite reference [14, 25, 24]. The output contains A, not a quantum return of C. It reports installed descriptions; it does not acquire an unknown continuum wavefront set.

Finite linear tests do not determine a general distribution, and therefore do not determine its wavefront set. For

example, for any fixed finite collection of tests one can choose more point-supported distributions than tests, at distinct regular points of a coordinate patch. Their evaluation map has a nonzero kernel. A nonzero combination in that kernel is singular but invisible to all the chosen tests. This is a coverage limitation in an unrestricted distribution class, not a construction of another Friedrichs covariance. The class-relative distinction is also emphasized in Oai282-A [26].

Evidence and unrelated analytic problems

All wavefront conclusions here are conclusions from the displayed representation and microlocal arguments. No numerical sample is promoted to an equality of wavefront sets.

A future numerical probe must retain its angular lifts, bundle phases, boundary-value side, test localizations, frequency directions, precision, cutoffs, raw residuals, and total error allowances. We retain the distinctions , , and used in XVI-O1-ST [24]. No numerical execution, compiler execution, independent rebuild, or proof-assistant check is claimed.

The apex-diffracted relation of this paper is not the aligned-null characteristic contraction of the separate XIV/XV programme. No joint characteristic graph map is constructed. The supplied XIV-I, XIV-J, and XV-B records retain their own domains and analytic obligations [27, 28, 29]. A global theorem for this free two-point kernel cannot discharge those different products.

Ledger update and conclusion

begingroup 1.12 toprule Object Status here Exact boundary

midrule endfirsthead toprule Object Status here Exact boundary

midrule endhead bottomrule endfoot

Massless Friedrichs spectral covariance on M Retained and globally analyzed The kernel and state prescription are ; compact-test infrared control is retained.

Local Hadamard form Constructed Lemma ; each regular developing neighbourhood.

Global ordinary two-point wavefront set Exactly determined Theorem , including all angular transitions. This is the V2 result for the specified state.

Strict diffraction Classified Absent exactly for $(\alpha, \beta) \in \mathscr{Q}$; otherwise the complete joint relation $\mathscr{D}^{+}_{\mathrm{ap}}$ occurs.

Positive frequency and causal-kernel orientation Constructed Theorem and Corollary , with the charged ordering .

Selected products and restrictions Constructed Same-orientation kernel products, equal-time pullback, and timelike-curve pullback; not arbitrary interacting products.

Global hyperbolicity or state uniqueness Not established Non-global-hyperbolicity remains. No uniqueness among all algebraic ground states is claimed.

Quantum state or field algebra across $\mathscr{W}$ **Open here** Neither the present wavefront theorem nor the one-point finite parts of I-B-VP-1 construct it.

Massive, non-Friedrichs, thermal, spinor, constrained, interacting cases **Open here** No transfer from the present exact massless covariance is asserted.

endgroup

The precise V2 update. For the fixed massless Friedrichs spectral ground state, V2 is resolved at the level of the exact ordinary wavefront set of the global two-point covariance on the regular host. This is stronger than a generic-pair containment and includes transition angles and exceptional quotient sectors. It is not a classification of every state or every extension.

V1 retains the prescribed-completion qualifications of I-B-VP-1 [18]. The other V-items retain their own extension, mass, thermal, apex-curvature, spinor, and interacting scopes. Here ``Open" means not resolved in the declared companion scope, not unknown throughout the literature.

No sourcing or programme-wide status change. The cone, character, Friedrichs condition, and spectral state are prescribed inputs. The theorem does not derive the metric, impose Einstein or Maxwell dynamics, solve

backreaction, or establish spacetime emergence [14, 19]. No OpenAI catalogue claim is a mathematical premise.

No standing version-specific status elsewhere is updated. Any O.1= entry remains so unless independently closed in its own stated scope; an independently reported subsector result is neither verified nor revoked here. XVI-N is not discharged, the downstream XV-B quantum-admission exit is not opened, and is not promoted by this paper [24, 25]. These are non-transfers, not corpus-wide impossibility claims.

Conclusion. Summing before continuation gives a global kernel whose only possible nongeometric branch point is the apex travel front. At its angular transition, reduced-variable pullback and the bisolution equations remove the spurious angular covectors that a moving-sector differentiation could create. The exact diffraction coefficient then supplies the lower bound, while finite-character projection identifies every nondiffractive sector. The resulting equality separates three facts: the covariance is locally Hadamard, its frequency orientation is globally controlled, and its global propagation relation need not be the relation of regular null geodesics alone.

appendix

Two checks on the transition argument

Why a null-set convention cannot control a wavefront

At $\delta \in \mathscr{B}_{\alpha}$, the geometric image set changes and the diffraction weight has its corresponding angular transition. Choosing a pointwise value on that angular set does not determine a distributional extension: distributions supported there are not ruled out by agreement off the set.

The proof instead uses , already identified with the spectral covariance before taking a boundary value. At $Q = -1$, every possible wavefront covector is constrained to

$$\lambda \, \mathrm{d}Q + \nu \, \mathrm{d}\delta.$$

Since

$$Q + 1 = \frac{F_{\mathrm{ap}}}{2rr},$$

its endpoint components are

$$\kappa = -\frac{\lambda T}{rr'}\,\mathrm{d}t + \frac{\lambda(r+r')}{rr'}\,\mathrm{d}r + \nu\,\mathrm{d}\varphi + \frac{\lambda Z}{rr'}\,\mathrm{d}z,$$

$$\kappa' = \frac{\lambda T}{rr'}\,\mathrm{d}t' + \frac{\lambda(r+r')}{rr'}\,\mathrm{d}r' - \nu\,\mathrm{d}\varphi' - \frac{\lambda Z}{rr'}\,\mathrm{d}z'.$$

Their characteristic equations are

$$p(X,\kappa) = \frac{\nu^2}{\alpha^2 r^2}, \qquad p(X',\kappa') = \frac{\nu^2}{\alpha^2 r'\hat{}2}.$$

Thus both force $\nu = 0$. This calculation works when one or two angular transition representatives coincide, and in particular when $1/\alpha$ is an integer. No separate endpoint prescription is needed.

Why the exceptional criterion is joint

The two conditions

$$\sin(\pi\beta/\alpha) = 0, \qquad \sin(\pi(1-\beta)/\alpha) = 0$$

are simultaneous conditions on the entire angular coefficient. They are stronger than cancellation at $\delta = 0$, or at any other chosen angular slice.

When they hold, every Bessel order is an integer:

$$\nu_n = |Nn + p|.$$

The finite-character heat projection then proves the exact finite-image covariance, not merely cancellation of a first diffractive asymptotic coefficient. When they do not hold, analyticity and closedness of the wavefront set prove the full $\mathscr{D}^{+}_{\mathrm{ap}}$ lower bound. These two different arguments are what make the exceptional classification exhaustive.

Declarations

Mathematical domain. All two-point distributions and all wavefront equalities are on the smooth regular

spacetime M, with $r, r' > 0$. No value of a quantum field on $\mathscr{W}$ is defined.

Dependency scope. The Friedrichs operator, bundle convention, and spectral identification are those stated explicitly in Section . The kernel summation, all-angle wavefront upper bound, diffraction coefficient, quotient criterion, and equality theorem are proved in this paper. Other companion citations are identified as analytical inputs or scoped comparisons; unquoted corpus material is not an additional premise.

Novelty scope. The paper claims a completed analysis for the stated companion problem. It does not claim invention of conical diffraction, Aharonov–Bohm diffraction, the Hadamard condition, spectral ground states, or microlocal product calculus.

Verification and submission scope. This is a single-file pdfLaTeX manuscript with an internal bibliography. No successful build, numerical experiment, independent referee verification, journal acceptance, or proof-assistant certification is asserted.

References

[1] J. S. Dowker, Quantum field theory on a cone, *Journal of Physics A: Mathematical and General* **10** (1977), 115–124. DOI: 10.1088/0305-4470/10/1/023. Historical cone-kernel context; the present wavefront proof uses the displayed summed covariance.

[2] B. Linet, Quantum field theory in the space-time of a cosmic string, *Physical Review D* **35** (1987), 536–539. DOI: 10.1103/PhysRevD.35.536. Cited for cosmic-string Euclidean Green functions and local subtraction, not for the global theorem proved here.

[3] J. Cheeger and M. Taylor, On the diffraction of waves by conical singularities. I, *Communications on Pure and Applied Mathematics* **35** (1982), 275–331. DOI: 10.1002/cpa.3160350302.

[4] J. Cheeger and M. Taylor, On the diffraction of waves by conical singularities. II, *Communications on Pure and Applied Mathematics* **35** (1982), 487–529. DOI: 10.1002/cpa.3160350403.

[5] R. Melrose and J. Wunsch, Propagation of singularities for the wave equation on conic manifolds, *Inventiones*

Mathematicae **156** (2004), 235–299. https://arxiv.org/abs/math/0111255. General conic propagation context; no unverified application of a compact-cone theorem to the present edge product is used.

[6] R. Melrose, A. Vasy, and J. Wunsch, *Propagation of singularities for the wave equation on edge manifolds*. https://arxiv.org/abs/math/0612750. Cited for the broader geometric/diffractive edge framework; the exact product-cone calculation here is explicit.

[7] M. Yang, *Diffraction of the Aharonov–Bohm Hamiltonian*. https://arxiv.org/abs/2008.04129. Cited for magnetic diffraction and its diffraction coefficient; its two-dimensional Hamiltonian is not identified without calculation with the four-dimensional covariance.

[8] M. J. Radzikowski, Micro-local approach to the Hadamard condition in quantum field theory on curved space-time, *Communications in Mathematical Physics* **179** (1996), 529–553. DOI: 10.1007/BF02100096. The globally hyperbolic equivalence theorem is a comparison, not a local-to-global theorem applied to this non-globally-hyperbolic host.

[9] L. Hörmander, *The Analysis of Linear Partial Differential Operators I: Distribution Theory and Fourier Analysis*, 2nd edition, Springer, 1990. Used for distributional boundary values, wavefront sets, pullbacks, products, and microlocal elliptic regularity.

[10] L. Hörmander, *The Analysis of Linear Partial Differential Operators IV: Fourier Integral Operators*, Springer, 1985. Used for propagation of singularities on the smooth regular host; no propagation theorem through an unspecified apex domain is invoked.

[11] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, 2nd edition, Cambridge University Press, 1944. Used for the Bessel heat integral and its Laplace transform.

[12] B. S. Kay and U. M. Studer, Boundary conditions for quantum mechanics on cones and fields around cosmic strings, *Communications in Mathematical Physics* **139** (1991), 103–139. DOI: 10.1007/BF02102731. Cited for the independent role of the apex self-adjoint extension.

Cheung companion records. The following references are unpublished manuscripts or supplied source excerpts, identified by version and source label. Citation does not assert an audit of an entire record, a successful compiler run, or formal verification.

[13] N. Cheung, *The Philosophy of Enter: Destructive Readout, Quantum Recovery, and the Refusal of Identification*, Cheung Ngo Medical Limited (2026). Framing essay; not a theorem premise. Programme note: https://enter.grok.me/enter.

[14] N. Cheung, *I-B-v1: A Finite Enter Spacetime: Calibrated Destructive Time Readout on a Prescribed Conical Product*, version 1, Enter Calculus Companion. Used: the prescribed regular host and worldsheet, thm:instrument, the Einstein comparator, and prop:not-source. https://enter.grok.me/browse/I-B. Companion record; not a Scholar-indexed journal article.

[15] N. Cheung, *I-B-G: Intrinsic and holonomy-twisted Green operators on a prescribed conical spacetime: Self-adjoint extensions, diffraction, finite propagation, and supported transfer from flat-completion laboratories*, version 2, October 2026. Supplied source I-B-G-v2.tex and quoted kernel passages, Enter Calculus Companion. Relevant labels: ass:host, prop:nonGH, thm:closed, thm:speed, cor:structure, thm:transfer, and ass:transfer-contract. Supported propagation is not covariance or state transfer. https://enter.grok.me/browse/I-B-G. Companion record; not a Scholar-indexed journal article.

[16] N. Cheung, *I-B-VP: Vacuum polarization of the Friedrichs ground state on a prescribed conical spacetime: Local Hadamard subtraction, holonomy, and the renormalized stress tensor*, version 2, October 2026. Supplied source I-B-VP-v2.tex, Enter Calculus Companion. Relevant labels: def:state, lem:infrared, prop:state, prop:euclidean, thm:analytic, thm:hadamard, thm:phi2, and prop:contour. Its V2 is the specific global two-point target addressed here. https://enter.grok.me/browse/I-B-VP. Companion record; not a Scholar-indexed journal article.

[17] N. Cheung, *I-B-VP-0: The holonomy-odd vacuum current of the charged Friedrichs ground state: A local observable on the prescribed conical Enter spacetime*, version 1, October 2026. Supplied source I-B-VP-0-v2.tex,

Enter Calculus Companion. Relevant labels: prop:kernel, def:current, rem:imaginary, thm:current, and the differentiated image–diffraction appendix. The current is a one-point observable, not the two-point wavefront relation proved here. https://enter.grok.me/browse/I-B-VP-0. Companion record; not a Scholar-indexed journal article.

[18] N. Cheung, *I-B-VP-1: Prescribed apex finite parts on a conical Enter spacetime: Wick square, stress tensor, and charged current: strong extension, Ward identities, and classified contact freedom*, version 1, October 2026. Supplied source I-B-VP-1-v1.tex, Enter Calculus Companion. Cited only to distinguish prescribed one-point extensions in an auxiliary smooth completion from a two-point state or field algebra across the apex. Its finite-part and contact-space theorems are not premises of the present wavefront proof. https://enter.grok.me/browse/I-B-VP-1. Companion record; not a Scholar-indexed journal article.

[19] N. Cheung, *X-A-ST*, version 1, Enter Calculus Companion. Used for the non-sourcing and supported-completion distinctions, including prop:noeinstein. https://enter.grok.me/browse/X-A-ST. Companion record; not a Scholar-indexed journal article.

[20] N. Cheung, *IX-A-ST*, version 1, Enter Calculus Companion. Supplied excerpts concerning finite destructive clock reporting and its distinction from a global continuum field theory. https://enter.grok.me/browse/IX-A-ST. Companion record; not a Scholar-indexed journal article.

[21] N. Cheung, *VII-H*, version 1, VII-H.tex, Enter Calculus Companion. Supplied free-field response and release excerpts. Methodological comparison for oriented products and the restriction of an ambiguity class, not a transferred conical product theorem. https://enter.grok.me/browse/VII-H. Companion record; not a Scholar-indexed journal article.

[22] N. Cheung, *XI-E*, version 1, XI-E.tex, Enter Calculus Companion. Supplied excerpts concerning its declared polynomial Hadamard domain and separate classical calculations. No conical interacting algebra or state selection is imported. https://enter.grok.me/browse/XI-E. Companion record; not a Scholar-indexed journal article.

[23] N. Cheung, *Em01-D*, version 2, Em01-D-v2.tex, Enter Calculus Companion. Supplied scope and

Fourier-certificate descriptions. Used only for the distinction between a finite-order bound and an actual wavefront theorem. https://enter.grok.me/browse/Em01-D. Companion record; not a Scholar-indexed journal article.

[24] N. Cheung, *XVI-O1-ST*, version 5, XVI-O1-ST-v5.tex, Enter Calculus Companion. Used for the complete supplied-report instrument, evidence classes, retained residuals, and execution boundaries. https://enter.grok.me/browse/XVI-O1-ST. Companion record; not a Scholar-indexed journal article.

[25] N. Cheung, *En09-A*, version 1, Enter Calculus Companion. Supplied destructive-clock and standing-boundary excerpts. https://enter.grok.me/browse/En09-A. Companion record; not a Scholar-indexed journal article.

[26] N. Cheung, *Oai282-A*, version 1, Enter Calculus Companion. Cited for the class-relative finite-test coverage distinction in the supplied companion material. No external catalogue theorem is a premise. https://enter.grok.me/browse/Oai282-A. Companion record; not a Scholar-indexed journal article.

[27] N. Cheung, *Landmark XIV-I*, version 1, XIV-I.tex, Enter Calculus Companion. Supplied excerpts: thm:I2, thm:I3, and thm:release. Internal amplitude construction and the separate unconstructed characteristic composition are recorded as non-transfer boundaries only. https://enter.grok.me/browse/XIV-I. Companion record; not a Scholar-indexed journal article.

[28] N. Cheung, *Landmark XIV-J*, version 1, XIV-J.tex, Enter Calculus Companion. Supplied excerpts: prop:descent, thm:reference, thm:J2, and cor:noadmission. Its supported-sector reference is not identified with the present apex-diffracted covariance. https://enter.grok.me/browse/XIV-J. Companion record; not a Scholar-indexed journal article.

[29] N. Cheung, *XV-B*, version 2, XV-B.tex, Enter Calculus Companion. Supplied relative-lifting, normal-jet, and release excerpts. Its technical existence and its distinct downstream quantum-admission exit remain separate; neither is discharged by the present free two-point theorem. https://enter.grok.me/browse/XV-B. Companion record; not a Scholar-indexed journal article.

[30] N. Cheung, *Black-Hole Information as an Access Problem: Enter, Quantum Recovery, and Complete Residuals*, figshare (2026). DOI: 10.6084/m9.figshare.34335024.v1. https://doi.org/10.6084/m9.figshare.34335024.v1.

[31] N. Cheung, *Operational Wormhole Certificates—Existence, Dimension-Dependent Scaling, Operational Functions, and Composition*, figshare (2026). DOI: 10.6084/m9.figshare.34549575.v1. https://doi.org/10.6084/m9.figshare.34549575.v1.

[32] N. Cheung, *Enter, Quantum Magic, and Quantum Gravity: Readout, Recovery, and the Limits of Geometric Response*, figshare (2026). DOI: 10.6084/m9.figshare.34335582.v1. https://doi.org/10.6084/m9.figshare.34335582.v1.

[33] N. Cheung, *Operational Consistency Does Not Select Lorentzian Geometry*, figshare (2026). DOI: 10.6084/m9.figshare.34441758.v1. https://doi.org/10.6084/m9.figshare.34441758.v1.

[34] N. Cheung, *Realization of Relational Time by Destructive Enter on a Clock Register*, figshare (2026). DOI: 10.6084/m9.figshare.34442907.v1. https://doi.org/10.6084/m9.figshare.34442907.v1.