I-B-VP-3-Ex1 Massive second-order insertions, defect renormalization, and controlled massless limits

Exact contact-sector formulas and a localized two-vertex construction on a prescribed conical spacetime

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Abstract

We investigate the proposed extension of the massive conical vacuum-polarization programme to second order in a quartic coupling. Three corrections are indispensable: a fixed extension with a negative transverse bound state cannot reach zero mass through its above-threshold ground-state regime; a mass gap does not remove vertex-collision divergences; and the order-λ^2 two-point correction of real $\lambda\varphi^4/4!$ theory is generally two-loop, not one-loop.

Two explicit constructions replace the unsupported general claim. First, we renormalize the repeated quadratic worldsheet insertion induced by the first-order apex contact. Its massive covariance, Wick square, and complete regular mixed stress reduce to convergent one-dimensional Macdonald integrals. Their compact-regular massless limits have an explicit error bound of order $m^2(1 + |\log m|)^3$. At second Born order the Wick square contains a radial logarithm, whereas the conformally improved massless stress has no subtraction-scale logarithm and equals

$$-\frac{g^2}{576\pi^4\alpha^3 r^4}\operatorname{diag}(1, 1, -3, 1).$$

The normalization $g = \lambda c/2$ identifies this as the specified order-$\lambda^2 c^2$ contact sector, not the full quartic correction.

Second, for compact regular interaction vertices separated from the observation region, we construct all three connected order-λ^2 real-scalar two-point contractions by explicit Taylor extensions of the squared and cubed

propagators. The resulting exterior numerical insertions have a controlled Friedrichs massless limit and local free-field tensor identities. We exhibit a regular bulk normalization freedom that cannot be classified by the three free apex-contact parameters.

The contribution is a convention-complete separation of infrared stabilization, collision renormalization, and massless continuation, with exact formulas for a nontrivial second-order apex-sensitive sector. General codimension-two coupling renormalization and interacting conical field theory are established subjects. The complete unswitched second-order stress and charged-current package, intersecting apex collisions, and a positive interacting state remain unconstructed. No finite Enter record acquires these continuum quantities, and no Einstein dynamics, backreaction, or spacetime emergence is inferred.

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Book expansion: two sectors and graph table

Keep visibly distinct: (A) repeated quadratic defect insertion; (B) compactly switched bulk vertices. toprule Sector Graph/contraction Collision prescription

midrule Quadratic defect As in the sector-(A) formulae Beside each coefficient

Switched bulk Compact switch fixed Not an adiabatic/apex limit

Massless estimate Switch held fixed ≠ adiabatic or apex limit

bottomrule

Opening boundaries and the corrected question

Enter and spacetime Enter

Cheung's Enter calculus characterizes a destructive finite measurement interface within a stated operational class [11]. The introductory *Philosophy of Enter* distinguishes a classical record, a quantum-return operation, an installed description, and a physical identification [12]. These distinctions guide interpretation; they are not assumptions in the Bessel or renormalization calculations.

For a calibrated finite clock C, surviving register A, and orthonormal basis $\{|\tau_j\rangle\}$, put

$$D_j = \langle \tau_j|_C \otimes I_A, \qquad \mathscr{M}(X) = \sum_j \mathrm{lvert}j, w_j\rangle\mathrm{langle}j, w_j|_R \otimes D_j X D_j^\dagger.$$

Then

$$\sum_j D_j^\dagger D_j = I_{CA}, \qquad \mathrm{Tr}_R\, \mathscr{M}(X) = \mathrm{Tr}_C\, X.$$

Expansion in clock matrix units proves both identities, including untouched-reference extensions. The output contains the specified residual A, not the measured quantum names C. This is the I-B/En09-A record–residual interface [13, thm:instrument] [14].

It does not acquire an unknown continuum graph coefficient, choose a coupling, prepare an interacting vacuum, or normalize an apex contact. Spacetime Enter places this interface on a separately prescribed geometry [13]. The geometric time axis is not consumed.

Scope boundary (Non-sourcing). The cone, extension, state prescription, masses, couplings, switches, and subtraction conventions are inputs. A conserved tensor is not thereby the source of its background. No Maxwell or Einstein equation is imposed by the formulas below. The I-B prop:not-source and X-A-ST prop:noeinstein distinctions remain unchanged [13, 22].

The supplied analytical baseline

The complete supplied source I-B-VP-3 constructs massive free ground states above the transverse bound-channel threshold, regular observables, the threshold logarithm, and the Friedrichs compact-regular massless limit [17]. The supplied I-B-VP-1 and its two corrective extensions distinguish free one-point finite parts, boundary operators, core diagnostics, and a first-order quartic tadpole [18, 19, 20].

Those results do not already define a second-order interaction. In particular:

1. a free Hadamard subtraction is not an extension of every power of a propagator;

2. convergence of free one-point functions is not a dominated convergence theorem for integrated graph products;

3. a finite-rank resolvent difference does not make a graph with several internal momenta a finite-rank or one-dimensional integral;

4. a free one-point contact classification is not a classification of interaction counterterms.

Self-interacting scalar fields on cones and codimension-two coupling renormalization have substantial prior literature [5, 6, 7]. The present paper claims neither their invention nor an all-orders completion.

Exact constructive scope

We release two different objects.

Sector D A neutral Friedrichs quadratic defect insertion through second Born order, on the ideal worldsheet, with its new relative-worldsheet extension specified explicitly. This computes the c^2 sector induced by the first-order contact coupling.

Sector B All connected two-vertex real-scalar two-point contractions for a compact Euclidean switch supported in a regular chart, with observations separated from the vertices. The internal collision extensions are fixed explicitly. This does not include vertices touching the apex or an adiabatic removal of the switch.

The letters D and B are organizational labels only. The sectors do not together constitute the full λ^2 apex interaction: mixed bulk–defect terms and joint apex collisions remain separate.

Host, extension threshold, and limits that are actually available

Prescribed geometry

Fix $0 < \alpha < 1$. The Lorentzian regular host is

$$M = \mathbb{R}_t \times (0,\infty)_r \times S^1_\varphi \times \mathbb{R}_z, \qquad g = -\mathrm{d}t^2 + \mathrm{d}r^2 + \alpha^2 r^2 \mathrm{d}\varphi^2 + \mathrm{d}z^2.$$

Its missing completion worldsheet is

$$\mathscr{W} = \mathbb{R}_t \times \{o\} \times \mathbb{R}_z.$$

For Euclidean calculations write

$$y = (\tau, z), \qquad \mathrm{d}V_E = \alpha r \, \mathrm{d}r \, \mathrm{d}\varphi \, \mathrm{d}^2 y.$$

The auxiliary smooth completion uses $x = (r\cos\varphi, r\sin\varphi)$, not developing coordinates, and density $\alpha \, \mathrm{d}^2 x \, \mathrm{d}^2 y$. The distribution $\delta_{\mathscr{W}}$ is normalized by

$$\langle \delta_{\mathscr{W}}, f \rangle = \int_{\mathbb{R}^2} f(y, 0) \, \mathrm{d}^2 y.$$

A product extension means a fixed transverse self-adjoint operator H_W, tensored with the ordinary longitudinal Laplacian, as in I-B-VP-3 [17]. Its parameters are independent of the mass and longitudinal momentum.

If its lowest transverse energy is $-\kappa_*^2$, set

$$B_{W,m} = H_W - \partial_z^2 + m^2.$$

Proposition (The admissible mass path). *The above-threshold free covariance*

$$W_{W,m}(T) = \frac{e^{-\mathrm{i}T\sqrt{B_{W,m}}}}{2\sqrt{B_{W,m}}}$$

has a positive spectral gap when $m > \kappa_*$.

For a fixed extension with $\kappa_* > 0$*, there is no sequence of above-threshold masses tending to zero. Changing the extension with m would be a different problem and would require a domain/state comparison theorem.*

For the Friedrichs extension, $\kappa_* = 0$*, and the massless path* $m\text{downarrow}\,0$ *is available. Its use in graph coefficients still requires graph-specific uniform estimates.*

Proof. Fourier transformation in z gives the spectral bands

$$m^2 - \kappa_j^2 + p_z^2$$

for transverse bound channels, and $[m^2, \infty)$ for the transverse continuum. The lowest band starts at $m^2 - \kappa_*^2$. The remaining assertions are immediate consequences of the strict inequality $m > \kappa_*$.

The threshold $m = \kappa_* > 0$ has the infrared obstruction identified in I-B-VP-3, even on regular compact tests [17, thm:threshold, thm:thresholdlaw]. Excluding the critical mass does not create a path around it to $m = 0$ at fixed extension.

Neutral Friedrichs Euclidean kernel

The positive Euclidean kernel used in both constructive sectors is

$$G_m(X,X') = \int_0^\infty \frac{e^{-m^2 s - |y-y'|^2/(4s)}}{4\pi s} \frac{e^{-(r^2+r'^2)/(4s)}}{4\pi\alpha s} \cdot \sum_{n\in\mathbb{Z}} e^{in(\varphi-\varphi')} I_{|n|/\alpha}\left(\frac{rr'}{2s}\right) \mathrm{d}s.$$

It solves

$$(-\Delta_E + m^2)G_m = \delta.$$

The free Wick square F_m is its diagonal remainder after subtracting the full massive flat kernel

$$P_m(\rho) = \frac{mK_1(m\rho)}{4\pi^2\rho}.$$

At $m = 0$, $P_0(\rho) = 1/(4\pi^2\rho^2)$.

For a regular developing product chart,

$$G_m(X,X') = P_m(|X-X'|) + H_m(X,X'),$$

with H_m smooth and locally analytic. The heat formula and local subtraction are the neutral Friedrichs specialization of I-B-VP-3 [17, thm:Hadamard].

Why mass stabilization is not second-order renormalization

Correct graph order

Proposition (Quartic two-point loop count). *A connected two-point graph with two quartic vertices and no counterterm vertices has three internal lines and two independent loops:*

$$I = 3, \qquad L = I - V + 1 = 2.$$

The two vertices cannot define the requested two-point graph merely by joining them with one line. A one-loop graph with two quartic vertices naturally has four external legs.

Proof. The eight quartic half-edges leave six internal half-edges after two external legs are attached, hence $I = 3$. Euler's connected graph formula gives $L = 2$. With four external legs, $I = 2$ and $L = 1$.

Closing the two external legs at a Wick-square insertion adds the corresponding observation contraction. Thus coupling order, loop order, and field degree must not be used interchangeably.

A regular collision survives every positive mass

Proposition (A mass gap does not remove bulk collision extensions). *In a regular flat chart, for every $m > 0$,*

$$G_m(u,v)^2 \sim \frac{1}{16\pi^4|u-v|^4}, \qquad G_m(u,v)^3 \sim \frac{1}{64\pi^6|u-v|^6}.$$

They are not locally integrable across $u = v$ in four relative dimensions. Their extension problems have degrees of divergence zero and two, respectively.

For a smooth relative test ψ, the squared leading kernel has logarithmic residue

$$\frac{1}{8\pi^2}\psi(0)\log(1/\varepsilon).$$

The cubed leading kernel has terms

$$\frac{\psi(0)}{64\pi^4\varepsilon^2} + \frac{\Delta\psi(0)}{256\pi^4}\log(1/\varepsilon),$$

in addition to lower singular terms from the subleading propagator expansion. A free coincidence subtraction does not specify these extensions.

Proof. The small-argument expansion of K_1 gives

$$P_m(\rho) = \frac{1}{4\pi^2\rho^2} + O(m^2|\log(m\rho)|).$$

The leading terms are therefore independent of the mass. The four-dimensional volume is $\rho^3 d\rho\, d\omega$, and

$$|S^3| = 2\pi^2, \qquad \int_{S^3} \omega_a\omega_b\, d\omega = \frac{\pi^2}{2}\delta_{ab}.$$

Insert the Taylor expansion of ψ. The displayed radial integrals are respectively $\int d\rho/\rho$, $\int d\rho/\rho^3$, and $\int d\rho/\rho$.

The extension framework is standard [1, 3]. The proposition is a direct counterexample to the plan's claim that the remaining second-order ultraviolet problems have already been removed by the free Hadamard subtraction.

Scope boundary (No universal obstruction is inferred). The failure of the proposed shortcut is not a theorem that second-order perturbation theory cannot be constructed. It identifies additional extension and normalization data. The following sections supply those data for two specified sectors, not for every apex graph.

Sector D: the massive worldsheet primitive

Bulk-to-worldsheet normalization

For the neutral Friedrichs kernel, the apex endpoint is

$$A_m(y;r) := G_m((y,r,\varphi),(0,o)) = \frac{1}{\alpha}\frac{mK_1(m\sqrt{r^2+|y|^2})}{4\pi^2\sqrt{r^2+|y|^2}}.$$

It is independent of φ. With Fourier convention

$$\hat{f}(k) = \int_{\mathbb{R}^2} e^{-\mathrm{i}k\cdot y} f(y)\,\mathrm{d}^2y, \qquad f(y) = \int_{\mathbb{R}^2} \frac{\mathrm{d}^2k}{(2\pi)^2} e^{\mathrm{i}k\cdot y}\hat{f}(k),$$

one has

$$\widehat{A}_m(k;r) = \frac{1}{2\pi\alpha}K_0(qr), \qquad q = \sqrt{|k|^2+m^2}.$$

Proof. Only the zero angular mode survives at an apex endpoint. The angular normalization gives $1/\alpha$ times the ordinary flat endpoint kernel. Fourier transformation in the two longitudinal variables leaves the decaying two-dimensional radial Green function, $K_0(qr)/(2\pi\alpha)$. Alternatively both identities follow by integrating the zero-mode term of .

For $r > 0$, this endpoint operation is supported by the boundary-field results of the supplied Ex1/Ex2 sources [19, 20]. It does not restrict the nonzero c-number finite part $sU_L + c\delta_{\mathscr{N}}$ to $\mathscr{W}$.

The new relative-worldsheet subtraction

A second quadratic insertion encounters the worldsheet kernel with both transverse endpoints at the apex. For separated $y \neq 0$, its Euclidean value is

$$\frac{1}{\alpha}\frac{mK_1(m|y|)}{4\pi^2|y|}.$$

It has the same $1/|y|^2$ singularity in two relative worldsheet dimensions for every $m \geq 0$.

Definition (Renormalized worldsheet primitive). Fix an inverse length $\mu > 0$. Define the tempered distribution T_m^μ by

$$\widehat{T_m^\mu}(k) = \frac{1}{4\pi\alpha}\log\frac{\mu^2}{|k|^2 + m^2}.$$

At $m = 0$, the locally integrable logarithm defines the same formula in $\mathscr{S}'(\mathbb{R}^2)$.

Theorem (Cutoff removal and the mass-independent local residue). *Let*

$$\widehat{T}_{m,\Lambda}(k) = \frac{1}{2\pi\alpha}\int_0^\Lambda \frac{\ell\,\mathrm{d}\ell}{\ell^2 + |k|^2 + m^2}.$$

Then

$$T_{m,\Lambda} - \frac{1}{2\pi\alpha}\log(\Lambda/\mu)\delta^{(2)} \to T_m^\mu \quad \text{in } \mathscr{S}'(\mathbb{R}^2).$$

For m in a bounded interval, the error is $O(\Lambda^{-2})$ in fixed Schwartz seminorm pairings. The coefficient of the logarithmic subtraction is independent of m.

For $y \neq 0$, $T_m^\mu(y)$ equals the separated worldsheet kernel above. Moreover,

$$\mu\partial_\mu T_m^\mu = \frac{1}{2\pi\alpha}\delta^{(2)}.$$

Proof. The cutoff integral is elementary:

$$\widehat{T}_{m,\Lambda}(k) = \frac{1}{4\pi\alpha}\log\frac{\Lambda^2 + |k|^2 + m^2}{|k|^2 + m^2}.$$

After subtracting the constant in , the difference from is

$$\frac{1}{4\pi\alpha}\log\left(1 + \frac{|k|^2 + m^2}{\Lambda^2}\right).$$

Its absolute value is bounded by $(|k|^2 + m^2)/(4\pi\alpha\Lambda^2)$. This proves the tempered convergence and its error estimate.

The inverse Fourier transform away from zero is identified by differentiating in m^2:

$$\partial_{m^2}\widehat{T}^{\mu}_{m} = -\frac{1}{4\pi\alpha}(|k|^2 + m^2)^{-1}.$$

The two-dimensional inverse transform is $-K_0(m|y|)/(8\pi^2\alpha)$, which is also the m^2-derivative of the stated separated kernel. The remaining mass-independent constant is fixed by the cutoff construction, modulo a delta at $y = 0$. The scale derivative follows immediately.

Remark (Established codimension-two mechanism). The logarithmic primitive and its coupling renormalization are the standard codimension-two mechanism studied by Goldberger–Wise and de Rham [6, 7]. The normalization here is fixed by –. No novelty is claimed for the existence of that mechanism.

Sector D: second Born covariance and regular observables

The prescribed perturbative sector

For a real scalar write the quadratic defect action convention as

$$S_{\mathrm{def}} = \frac{g}{2}\int_{\mathscr{W}} \varphi^2.$$

The following are numerical perturbative kernels:

$$C_{1,m}(X, X') = -\int_{\mathbb{R}^2} A_m(y - y_1; r)A_m(y_1 - y'; r')\,\mathrm{d}^2y_1,$$

$$C^{\mu}_{2,m}(X, X') = \int_{\mathbb{R}^2\times\mathbb{R}^2} A_m(y - y_1; r)T^{\mu}_m(y_1 - y_2)A_m(y_2 - y'; r')\,\mathrm{d}^2y_1\mathrm{d}^2y_2.$$

The second expression is defined by its Fourier pairing. The covariance correction through this order is

$$\delta G^{\mathrm{D}}_m = gC_{1,m} + g^2C^{\mu}_{2,m}.$$

Theorem (Exact massive defect-sector formula). *For regular endpoints,*

$$\delta G_m^{\mathrm{D}}(X,X') = \int_{\mathbb{R}^2} \frac{\mathrm{d}^2k}{(2\pi)^2} e^{\mathrm{i}k\cdot(y-y')} \frac{K_0(qr)K_0(qr')}{4\pi^2\alpha^2} \left[-g + \frac{g^2}{4\pi\alpha} \log\frac{\mu^2}{q^2} \right].$$

It is smooth at every regular diagonal point and is a massive Euclidean bisolution off $\mathscr{W}$. It has a local Lorentzian analytic continuation whenever $|t-t'| < r + r'$, after restricting to a smaller compact pair neighbourhood if necessary.

Replacing T_m^{μ} by the subtracted cutoff primitive in Theorem changes every fixed regular endpoint jet by $O(\Lambda^{-2})$.

This is a finite-order insertion, not a claim that $G_m + \delta G_m^{\mathrm{D}}$ is an exact positive covariance or a resummed self-adjoint extension.

Proof. Fourier transformation of gives . At high $|k|$, the product of Macdonald functions is a polynomial factor times $e^{-|k|(r+r')}$. At small $|k|$, the worst massless behavior is a finite power of $|\log|k||$, integrable with measure $|k|\mathrm{d}|k|$. All fixed regular derivatives are therefore allowed.

The radial equation for each $K_0(qr)$ proves the endpoint field equations. The same high-momentum bound permits complex time differences with imaginary part smaller than $r + r'$. The cutoff error is a polynomial in k, times $O(\Lambda^{-2})$, against exponentially decaying endpoint factors. This proves the jet estimate.

Connection with the free apex contact

The first-order tadpole contact of the supplied Ex1/Ex2 convention contributes

$$-\frac{\lambda c}{2} \int_{\mathscr{W}} G_m(X,u) G_m(u,X') \, \mathrm{d}^2u.$$

Thus

$$g = \frac{\lambda c}{2}.$$

The term $g^2 C_{2,m}^{\mu}$ is consequently the specified order-$\lambda^2 c^2$ repeated-contact contribution.

It does not include the sunset, nested bulk tadpoles, mixed bulk–defect graphs, or a full second-order

renormalization of the original quartic interaction.

Moment formulas

Set

$$w_g(q) = -g + \frac{g^2}{4\pi\alpha}\log\frac{\mu^2}{q^2}, \qquad c_\alpha = \frac{1}{8\pi^3\alpha^2}.$$

Define

$$F_m^{\mathrm{D}}(r) = c_\alpha \int_m^\infty q\, w_g(q) K_0(qr)^2\, \mathrm{d}q,$$

$$L_m^{\mathrm{D}}(r) = c_\alpha \int_m^\infty \frac{q(q^2-m^2)}{2} w_g(q) K_0(qr)^2\, \mathrm{d}q,$$

$$R_m^{\mathrm{D}}(r) = c_\alpha \int_m^\infty q^3 w_g(q) K_1(qr)^2\, \mathrm{d}q, \qquad V_m^{\mathrm{D}} = 0.$$

These formulas include the factors g, g^2. Let

$$D_m^{\mathrm{D}} = (F_m^{\mathrm{D}})'' + r^{-1}(F_m^{\mathrm{D}})'.$$

Theorem (Complete regular real-scalar stress in Sector D). *In order* (t, r, φ, z)*, the nonzero mixed components are*

$$T^t{}_{t,\mathrm{D}} = T^z{}_{z,\mathrm{D}} = L_m^{\mathrm{D}} + \left(\xi - \frac{1}{4}\right) D_m^{\mathrm{D}},$$

$$T^r{}_{r,\mathrm{D}} = R_m^{\mathrm{D}} - \xi (F_m^{\mathrm{D}})'' + \left(\xi - \frac{1}{4}\right) D_m^{\mathrm{D}},$$

$$T^\varphi{}_{\varphi,\mathrm{D}} = -\xi \frac{(F_m^{\mathrm{D}})'}{r} + \left(\xi - \frac{1}{4}\right) D_m^{\mathrm{D}}.$$

They obey on $r > 0$

$$\nabla_\mu T^\mu{}_{\nu,\mathrm{D}} = 0,$$

$$T^\mu{}_{\mu,\mathrm{D}} = 3\left(\xi - \frac{1}{6}\right) \mathscr{D}_r F_m^{\mathrm{D}} - m^2 F_m^{\mathrm{D}},$$

$$T_{\mu\nu,\mathrm{D}}(\xi) - T_{\mu\nu,\mathrm{D}}(1/6) = \left(\xi - \frac{1}{6}\right)(g_{\mu\nu}\mathrm{Box}_g - \nabla_\mu\nabla_\nu) F_m^{\mathrm{D}}.$$

All displayed integrals and fixed regular radial derivatives are absolutely convergent.

For an untwisted complex scalar with defect $g\int_{\mathscr{M}} \Phi^\dagger\Phi$*, the Wick square* $\langle\Phi^\dagger\Phi\rangle$ *has the same formula, the stress is twice , and the current correction vanishes.*

Proof. The angular zero mode gives $V = 0$. At equal Euclidean longitudinal endpoints, opposite time derivatives or opposite z derivatives average k_τ^2 or k_z^2 to $(q^2 - m^2)/2$. Radial derivatives give $q^2K_1^2$. This proves –.

For a smooth real massive bisolution correction C, put

$$F = [C], \qquad A_{\mu\nu} = [\nabla_\mu \nabla_{\nu'} C].$$

The regular real-scalar stress reduces to

$$T_{\mu\nu} = A_{\mu\nu} - \xi\nabla_\mu\nabla_\nu F + \left(\xi - \frac{1}{4}\right) g_{\mu\nu}\mathrm{Box}_g F.$$

The endpoint equations imply

$$A^\mu{}_\mu = \frac{1}{2}\mathrm{Box}_g F - m^2 F, \qquad \nabla^\mu A_{\mu\nu} = \frac{1}{4}\nabla_\nu \mathrm{Box}_g F.$$

These identities prove conservation, trace, and improvement. For a radial F, the mixed transverse Hessian entries are F'' and F'/r, giving .

The complex untwisted quadratic perturbation treats its two real components equally. Its kernel is real and even in the endpoint difference, so the antisymmetric complex first jet, hence the current, vanishes.

Scope boundary (Current scope and non-sourcing). The real neutral theory has no charged Noether current. The zero complex current statement above is only for the untwisted, charge-conjugation-invariant quadratic sector. It is not an evaluation of the order-λ^2 current of a twisted $(\Phi^\dagger\Phi)^2$ theory.

In a nontrivial Friedrichs holonomy sector, the bulk-derived unweighted apex endpoint vanishes [19, 20]; this removes this particular endpoint contact sector, not the ordinary bulk circulating current. No non-diagonal scalar matrix or gauge-field domain is imported.

The conserved stress remains a response on a prescribed background, not an Einstein source law [13, 22].

Sector D: exact massless formulas and quantitative convergence

Write

$$F_m^{\mathrm{D}} = gF_{1,m} + g^2F_{2,m}, \qquad T_{\mathrm{D}} = gT_1 + g^2T_2.$$

Define

$$\mathscr{L}(r) = \log(\mu r/2) + \gamma_E + 1, \qquad C_\alpha = \frac{1}{32\pi^4\alpha^3}.$$

Theorem (Massless defect-sector coefficients). *For $r > 0$,*

$$F_{1,0}(r) = -\frac{1}{16\pi^3\alpha^2r^2},$$

$$F_{2,0}(r) = \frac{C_\alpha}{r^2}\mathscr{L}(r).$$

The real-scalar first-order stress is

$$T^\mu_{1,0\nu} = -\frac{1}{16\pi^3\alpha^2r^4}\mathrm{diag}\left(4\xi - \frac{2}{3}, \frac{1}{3} - 2\xi, 6\xi - 1, 4\xi - \frac{2}{3}\right).$$

The second-order coefficients are

$$T^t_{t,2,0} = T^z_{z,2,0} = \frac{C_\alpha}{r^4}\left[\left(4\xi - \frac{2}{3}\right)\mathscr{L} - 4\xi + \frac{11}{18}\right],$$

$$T^r_{r,2,0} = \frac{C_\alpha}{r^4}\left[\left(\frac{1}{3} - 2\xi\right)\mathscr{L} + \xi - \frac{2}{9}\right],$$

$$T^\varphi_{\varphi,2,0} = \frac{C_\alpha}{r^4}[(6\xi - 1)\mathscr{L} + 1 - 5\xi].$$

At conformal coupling,

$$T_{1,0} = 0, \qquad g^2T^\mu_{2,0\nu} = -\frac{g^2}{576\pi^4\alpha^3r^4}\mathrm{diag}(1, 1, -3, 1).$$

Thus the Wick square has a scale-dependent logarithm at this order, whereas the conformal regular stress does

not.

Proof. The product Mellin identity is

$$\int_0^\infty x^{s-1}K_a(x)K_b(x)\,\mathrm{d}x = \frac{2^{s-3}}{\Gamma(s)}\prod_{\varepsilon,\delta=\pm1}\Gamma\Big(\frac{s+\varepsilon a+\delta b}{2}\Big), \quad \operatorname{Re} s > |a|+|b|.$$

It follows from the two Schwinger representations and a beta integral; the classical Bessel integral framework is recorded in [10].

The required values and logarithmic derivatives are

$$\int_0^\infty xK_0^2\,\mathrm{d}x = \frac{1}{2},$$

$$\int_0^\infty xK_0^2\log x\,\mathrm{d}x = \frac{1}{2}(\log 2-\gamma_E-1),$$

$$\int_0^\infty x^3K_0^2\,\mathrm{d}x = \frac{1}{3},$$

$$\int_0^\infty x^3K_0^2\log x\,\mathrm{d}x = \frac{1}{3}(\log 2-\gamma_E+\tfrac{1}{6}),$$

$$\int_0^\infty x^3K_1^2\,\mathrm{d}x = \frac{2}{3},$$

$$\int_0^\infty x^3K_1^2\log x\,\mathrm{d}x = \frac{2}{3}(\log 2-\gamma_E-\tfrac{1}{12}).$$

Substitution $x = qr$ gives –. For the second-order moments,

$$L_{2,0} = \frac{C_\alpha}{r^4}\Big(\frac{\mathscr{L}}{3}-\frac{7}{18}\Big), \qquad R_{2,0} = \frac{C_\alpha}{r^4}\Big(\frac{4\mathscr{L}}{3}-\frac{11}{9}\Big).$$

Also

$$F'_{2,0} = \frac{C_\alpha}{r^3}(1-2\mathscr{L}), \quad F''_{2,0} = \frac{C_\alpha}{r^4}(6\mathscr{L}-5), \quad \mathscr{L}_r F_{2,0} = \frac{4C_\alpha}{r^4}(\mathscr{L}-1).$$

Insert these into . At $\xi = 1/6$, all logarithmic stress coefficients vanish, giving .

Theorem (Controlled compact-regular massless limit). *Fix* $0 < r_0 < r_1 < \infty$, *a reference length* $\ell > 0$, *and a derivative order k. For* $0 < m\ell \le 1/2$, *the Sector D coefficients satisfy*

$$\lVert F_{j,m} - F_{j,0} \rVert_{C^k([r_0,r_1])} + \lVert T_{j,m} - T_{j,0} \rVert_{C^k([r_0,r_1])} \le C_{k,r_0,r_1,\ell,\mu} (m\ell)^2 (1 + |\log(m\ell)|)^3, \quad j = 1,2.$$

The same type of estimate holds for every fixed endpoint jet on compact pair neighbourhoods strictly inside the local continuation domain of Theorem .

No threshold is crossed: the reference extension in this theorem is Friedrichs.

Proof. In , the mass dependence is the lower integration limit. Uniformly for $r \in [r_0, r_1]$,

$$K_0(qr) = O(1 + |\log q|), \qquad K_1(qr) = O(q^{-1})$$

at small q. Fixed radial derivatives of $K_0(qr)$ are no worse in momentum power; positive derivative order removes rather than increases its leading logarithm.

The missing interval $0 < q < m$ in the second-order Wick square is therefore bounded by

$$C\int_0^m q(1 + |\log q|)^3 \, dq \le Cm^2(1 + |\log m|)^3.$$

The radial moment has the better integrand $O(q(1 + |\log q|))$. The longitudinal moment has, in addition, the factor $q^2 - m^2$. Its explicit m^2-term is bounded by a convergent massless moment, while its omitted low-energy interval is smaller than the stated bound. The stress follows by finite radial differentiation.

For separated endpoints, angular integration in longitudinal momentum gives a J_0 factor with argument $\sqrt{q^2 - m^2}|y - y'|$. Its difference from the massless factor, and the corresponding fixed derivatives, are controlled by m^2 times a polynomial on compact endpoint sets. The high-energy exponential margin remains uniform on compact subsets of the continuation domain. The same low-energy estimate completes the proof.

Scale changes and the first-order contact coordinate

Proposition (Finite scale reparametrization through second order). *For* $\mu' > 0$,

$$T_m^{\mu'} - T_m^{\mu} = \frac{1}{2\pi\alpha} \log(\mu'/\mu)\delta^{(2)}.$$

Consequently the Born correction is unchanged through order g^2 if

$$g(\mu') = g(\mu) + \frac{g(\mu)^2}{2\pi\alpha}\log(\mu'/\mu) + O(g^3).$$

This agrees with the established codimension-two quadratic coupling flow [6, 7]. It is not an all-orders renormalization theorem here.

Proof. The primitive shift increases $C_{2,m}$ by

$$\frac{1}{2\pi\alpha}\log(\mu'/\mu)\int_{\mathscr{W}} A_m A_m.$$

The corresponding change in the first Born term is the negative of this expression under . Terms from changing g^2 are order g^3.

A finite massless limit of these coefficients is not a positive exact massless point-interaction vacuum. A fixed finite logarithmic transverse realization has a negative bound scale, as recorded in I-B-VP-3 [17, 8, 9]. Proposition still applies to that exact realization. The perturbative calculation does not revoke it.

Sector B: all two-vertex real-scalar contractions on a regular support

Support and prescription are fixed first

Let $\mathscr{B}$ be a convex Euclidean developing chart whose closure stays away from the apex. Choose

$$\theta \in C_c^\infty(\mathscr{B};\mathbb{R})$$

as the quartic switch. Let $\mathscr{O}$ be a regular observation region whose closure has positive distance from $\operatorname{supp}\theta$.

For internal vertices u, v, write in the developing coordinates

$$s = \frac{u+v}{2}, \qquad w = u - v, \qquad Q_m(s,w) = G_m(s + w/2, s - w/2).$$

Choose an even smooth cutoff $\chi(w)$, equal to one near zero, with support small enough that the endpoint pairs produced by the following subtraction remain in a slightly larger regular chart. All center supports lie in a fixed

compact subset of that chart.

The switch, chart, and cutoff are independent of m.

Definition (Explicit internal collision extensions). For $k = 2, 3$, put $\omega_k = 2k - 4$. For a smooth compact two-vertex test $\psi(s, w)$, define

$$\langle \mathcal{R}_{k,m}, \psi \rangle = \int \mathrm{d}^4 s \, \mathrm{d}^4 w \, Q_m(s, w)^k [\psi(s, w) - \chi(w) \sum_{|\gamma| \le \omega_k} \frac{w^\gamma}{\gamma!} \partial_w^\gamma \psi(s, 0)].$$

No additional finite delta jets are added in this prescription. The test in this formula includes all switches, tadpole factors, and external propagator factors relevant to its graph.

Lemma (Absolute convergence and normalization freedom). *Equation is an absolutely convergent integral and defines an extension of $G_m(u, v)^k$ across $u = v$. It is symmetric in u, v and has relative scaling degree at most $2k$.*

Changing the cutoff or adding another extension with the same scaling bound changes it by

$$\sum_{|\gamma| \le \omega_k} a_\gamma(s) \partial_w^\gamma \delta^{(4)}(w),$$

with the appropriate regularity and symmetry constraints. For an endpoint-exchange-symmetric prescription, the odd normal jets are absent.

Proof. Near $w = 0$, $Q_m^k = O(|w|^{-2k})$, while the square bracket is $O(|w|^{\omega_k + 1})$. Including four-dimensional volume gives a bounded radial integrand:

$$|w|^{3 - 2k + \omega_k + 1} = |w|^0.$$

Away from zero the integral is over a compact support. The symmetry follows from the symmetry of G_m and evenness of χ.

The Taylor-subtraction estimate gives the stated scaling degree. A supported difference at the diagonal with that degree is a normal delta jet through order $2k - 4$. This is also the usual scaling-degree extension statement [1, 2].

The complete Wick-contraction list in this support class

Normal ordering the local quartic vertex with the same free massive Minkowski reference gives, in free-state Wick coordinates,

$$:\varphi^4:_{P_m} = :\varphi^4:_{W_m} + 6F_m :\varphi^2:_{W_m} + 3F_m^2.$$

This identity fixes the tadpole convention in the following numerical expansion.

Define the connected two-point correction by

$$G^{\theta}_{\mathrm{conn},m} = G_m + \lambda C^{\theta}_{1,m} + \lambda^2 C^{\theta}_{2,m} + O(\lambda^3).$$

No factorial is absorbed into $C^{\theta}_{2,m}$.

The first coefficient is

$$C^{\theta}_{1,m}(x,x') = -\frac{1}{2}\int \theta(u) F_m(u) G_m(x,u) G_m(u,x')\,\mathrm{d}V_u.$$

Theorem (All connected order-λ^2 two-point contractions). *For the prescribed real $\lambda\varphi^4/4!$ numerical expansion and supports above,*

$$C^{\theta}_{2,m} = C_{\mathrm{ch},m} + C_{\mathrm{sun},m} + C_{\mathrm{sn},m},$$

where

$$C_{\mathrm{ch},m}(x,x') = \frac{1}{4}\int \theta(u)\theta(v) F_m(u) F_m(v) \cdot G_m(x,u) G_m(u,v) G_m(v,x')\,\mathrm{d}V_u \mathrm{d}V_v,$$

$$C_{\mathrm{sun},m}(x,x') = \frac{1}{6}\left\langle \mathcal{R}_{3,m}, \theta(u)\theta(v) G_m(x,u) G_m(v,x') \right\rangle,$$

$$C_{\mathrm{sn},m}(x,x') = \frac{1}{4}\left\langle \mathcal{R}_{2,m}, \theta(u)\theta(v) F_m(v) G_m(x,u) G_m(u,x') \right\rangle.$$

Here $\mathcal{R}_{k,m}$ is understood in the center-relative coordinates of Definition . The chain integral is already absolutely convergent. The other two expressions are absolutely convergent Taylor-subtracted integrals.

The resulting $C^{\theta}_{2,m}$ is smooth on $\mathcal{O}\times\mathcal{O}$, including its diagonal, and is a Euclidean massive bisolution in its external variables there.

Proof. Expand the normalized Gaussian expectation to order λ^2. Disconnected vacuum components cancel against its denominator.

With two labelled external legs, the connected possibilities are:

1. one external leg at each vertex, one bridge, and one self-contraction at each vertex;

2. one external leg at each vertex and three bridges;

3. both external legs at one vertex, two bridges to the other vertex, and one self-contraction at that other vertex.

Their pairing counts, including the assignment of vertices, are $288, 192, 288$. Division by $2!\,(4!)^2 = 1152$ gives $1/4, 1/6, 1/4$, respectively. The first-order count is $12/4! = 1/2$, with its negative Euclidean expansion sign. Replacing each self-contraction by F_m and extending the squared and cubed bridges gives –.

In the chain, the only internal collision factor is $G_m(u,v) = O(|u-v|^{-2})$, locally integrable in four relative dimensions. For the other graphs use Lemma . All external propagator factors and their derivatives are smooth on the declared separated supports. Their derivatives are bounded test seminorms for $\mathcal{R}_{k,m}$. This proves smoothness.

Applying $-\Delta_x + m^2$ to an external factor gives $\delta(x-u)$ or $\delta(x-v)$. It vanishes on $\mathcal{O}$, since every original vertex test has the compact switch. The same remains true after taking its collision jets. Thus the external equations hold.

Remark (What this formula does and does not define). Equations – define a complete numerical contraction coefficient for this localized two-point problem. They are not asserted to furnish a family of time-ordered products satisfying every interacting normalization condition. In particular, no interacting state positivity or global real-time identification is inferred from a finite Euclidean coefficient.

Sector B: a graph-specific Friedrichs massless limit

Fix a length $\ell > 0$ and define, for $0 < m\ell < 1/2$,

$$\varepsilon_{\mathrm{mass}}(m) = (m\ell)^2(1 + |\log(m\ell)|).$$

Lemma (Kernel estimates needed for the graph limit). *On the fixed regular chart and center supports above, for every fixed center-derivative order a,*

$$|\partial_s^a Q_m(s,w)| \le C_a|w|^{-2},$$

$$|\partial_s^a(Q_m - Q_0)(s,w)| \le C_a m^2(1 + |\log(m\ell)| + |\log(|w|/\ell)|)$$

for sufficiently small nonzero w. On separated regular compact pair sets, $G_m - G_0 = O_{C^k}(\varepsilon_{\mathrm{mass}}(m))$. Moreover,

$$F_m - F_0 = O_{C^k}(\varepsilon_{\mathrm{mass}}(m))$$

on regular compact sets.

For $k = 2, 3$, the fixed Taylor prescription satisfies

$$\left|\langle \mathcal{R}_{k,m} - \mathcal{R}_{k,0}, \psi\rangle\right| \le C\,\varepsilon_{\mathrm{mass}}(m)\,p_{\omega_k+1}(\psi)$$

on each fixed compact test-support stage.

Proof. On regular compact sets, the four-dimensional heat kernel in , with any fixed center derivatives, is $O(s^{-2})$ for large s. This follows from the small-argument Bessel expansion: the neutral zero mode is the slowest term. Interior localization gives the direct flat heat kernel plus an exponentially small short-time remainder.

Split the proper-time difference at $s = \ell^2$ and $s = m^{-2}$. The long-time part is bounded by

$$C\int_{\ell^2}^{\infty}(1 - e^{-m^2 s})s^{-2}\,ds \le Cm^2(1 + |\log(m\ell)|).$$

At short time the direct term gives the usual $m^2\log|w|$ bound, equivalently obtained from the small-argument expansion of P_m. The localized remainder is smooth with the stated mass rate. Center derivatives kill the center-independent flat singularity and satisfy the same sufficient bounds. These observations prove , the separated estimate, and . They refine, for this compact chart, the free convergence statement of I-B-VP-3 [17].

For the final assertion,

$$Q_m^k - Q_0^k = (Q_m - Q_0) \sum_{j=0}^{k-1} Q_m^j Q_0^{k-1-j}.$$

Its absolute value is bounded by

$$C m^2 (1 + |\log m| + |\log |w||) |w|^{-2(k-1)}.$$

The Taylor remainder supplies $|w|^{\omega_k + 1}$. After four-dimensional radial integration the small-w power is

$$3 + \omega_k + 1 - 2(k-1) = 2,$$

which is integrable with the logarithm. The rest of the support is separated and smooth. This proves .

Theorem (Controlled massless limit of the localized two-vertex coefficient). *For the fixed switch, fixed collision prescription, and separated observation region,*

$$\mathrm{lVertC_}j,m^{\theta} - C_{j,0}^{\theta} \|_{C^k(K \times K)} \le C_{K,k} \varepsilon_{\mathrm{mass}}(m), \qquad j = 1, 2,$$

*for every compact K*Subset $\mathscr{O}$ *and finite k.*

*Its diagonal Wick-square coefficients and every fixed external differential contraction have the same compact-regular rate. The limit is independent of the path m*downarrow 0 *within the Friedrichs family.*

This theorem does not remove the compact switch, move vertices onto $\mathscr{W}$*, or identify its finite collision normalizations with those of another scheme.*

Proof. All external factors have the separated estimates of Lemma . The tadpoles have . For the chain, expand the difference of its finitely many factors and use the integrable internal singularity. For the other two graphs, write

$$\langle \mathcal{R}_{k,m}, \psi_m \rangle - \langle \mathcal{R}_{k,0}, \psi_0 \rangle = \langle \mathcal{R}_{k,m} - \mathcal{R}_{k,0}, \psi_0 \rangle + \langle \mathcal{R}_{k,m}, \psi_m - \psi_0 \rangle.$$

Equation controls the first term. The uniform Taylor-extension seminorm estimate and the separated/tadpole estimates control the second. The same argument applies after fixed external derivatives. Restriction to the external diagonal is continuous in these smooth norms.

Remark (Why free compact convergence was insufficient by itself). The proof uses fixed internal supports,

mass-independent Taylor subtractions, and an integrable graph-specific majorant. The free theorem alone supplies none of these three conditions for an unswitched apex graph. There is no interchange of an unproved adiabatic limit, an apex limit, and a massless limit.

Exterior tensor identities and the interacting Ward boundary

Local continuation of Sector B

For a real local Lorentzian tensor interpretation of the Sector B numerical insertion, impose the following additional condition:

$$\theta(-\tau, \mathbf{x}) = \theta(\tau, \mathbf{x}),$$

and choose the collision cutoff invariant under simultaneous Euclidean time reflection. Take the observation chart around $\tau = 0$, with its spatial points separated from the spatial projection of the vertex support.

The external Euclidean bisolution is jointly analytic in that observation chart, by analytic elliptic regularity. Its simultaneous time-reflection symmetry implies that its Taylor series has even total degree in the two time coordinates. Substituting $\tau = \mathrm{i}t, \tau' = \mathrm{i}t'$ on a sufficiently small complex neighbourhood therefore gives a real symmetric smooth Lorentzian bisolution $S_{j,m}$. This is a local continuation of the numerical coefficient, not a construction of an interacting global state.

Proposition (Exterior free tensor identities). *For $S_{j,m}$, define*

$$F^{\mathrm{B}}_{j,m} = [S_{j,m}], \qquad A_{\mu\nu,j} = [\nabla_\mu \nabla_{\nu'} S_{j,m}],$$

and define $T^{\mathrm{B}}_{\mu\nu,j}$ by . Then

$$\nabla^\mu T^{\mathrm{B}}_{\mu\nu,j} = 0,$$

$$T^{\mu\mathrm{B}}_{\mu,j} = 3\left(\xi - \frac{1}{6}\right)\mathrm{Box}_g F^{\mathrm{B}}_{j,m} - m^2 F^{\mathrm{B}}_{j,m}.$$

The improvement identity holds with the same scalar coefficient. These tensors inherit the massless rate on smaller

compact continuation neighbourhoods.

No charged-current coefficient is constructed by this real-scalar calculation.

Proof. The proof of Theorem applies to any smooth symmetric massive bisolution on a regular flat chart. The continuation is locally uniform with its derivatives; its mass estimates follow either from the holomorphic kernel bounds or the local analytic elliptic estimates on a fixed slightly larger observation chart.

Scope boundary (Non-sourcing). Equation is a local identity of a specified numerical insertion outside the interaction support. It does not determine the background or its metric response [13, 22].

Why these are not full interacting Ward identities

For a classical switched real quartic action on a regular flat region, the full stress includes

$$-g_{\mu\nu}\frac{\lambda\theta}{4!}\varphi^4.$$

Using the interacting field equation gives

$$\nabla^{\mu}T^{\mathrm{full}}_{\mu\nu} = -\frac{\lambda}{4!}(\partial_{\nu}\theta)\varphi^4.$$

Thus a switch can exchange stress with the field. Applying the free bisolution proof inside its support would be incorrect.

At the quantum level, conservation and trace identities require compatible normalizations of time-ordered products, the field equation, the interaction stress, and composite insertions. The trace can also contain interaction renormalization terms. Hollands–Wald prove the corresponding smooth-background compatibility under additional normalization conditions [4]. Their theorem is not automatically a theorem through the present conical apex.

The spin-1 and spin-2 source-test intertwiners of I-B-G-S-v2 concern different gauge and ghost domains [21]. They do not establish for an interacting scalar graph or select its counterterms.

Counterterms, contacts, and why the three-parameter transfer fails

Proposition (A regular bulk normalization changes the observable off the apex). *For $m > 0$, change the cubic bridge extension by*

$$\mathcal{R}_{3,m} \longmapsto \mathcal{R}_{3,m} + a\, m^2 \delta^{(4)}(u - v), \qquad a \in \mathbb{R}.$$

This respects endpoint-exchange symmetry and the relative scaling bound. It changes the sunset coefficient by

$$\Delta C_{\mathrm{sun},m}(x, x') = \frac{a m^2}{6} \int \theta(u)^2 G_m(x, u) G_m(u, x')\, \mathrm{d}V_u.$$

For nonzero θ, $a > 0$, and a regular external point $x = x'$, this change is strictly positive.

It cannot be represented by a c-number distribution supported on $\mathscr{W}$.

Proof. A delta at the four-dimensional collision has scaling degree four; multiplication by m^2 has the required engineering dimension for the cubed propagator. Substitution in gives . The neutral Friedrichs Euclidean kernel is positive, so its diagonal external integral is positive. A worldsheet-supported c-number distribution vanishes on every regular external test, while this function does not.

The free classification of I-B-VP-1 concerns simultaneous extensions of already fixed one-point functions, under specified symmetry, scaling, and Ward assumptions [18, thm:classification]. It is not contradicted by Proposition ; its object is different.

At second order one must distinguish at least:

1. regular bulk collision normalizations;

2. observation-composite normalizations;

3. couplings supported on $\mathscr{W}$;

4. extensions where a bulk collision meets $\mathscr{W}$;

5. purely c-number one-point completion contacts. The second-order worldsheet parameter μ in Definition belongs

to item 3, not merely to item 5.

The corrective Ex1/Ex2 analysis also shows why broad Killing–homothety uniqueness cannot be inferred from noninteger leading powers alone [19, 20]. No full non-diagonal contact count is imported here.

Stopping condition (The complete apex graph remains unconstructed). The full unswitched order-λ^2 coefficient for vertices approaching $\mathscr{W}$ requires joint extensions at intersecting loci:

$$u = v, \qquad u \in \mathscr{W}, \qquad v \in \mathscr{W},$$

and their intersections, together with the observation diagonals when the observation meets the interaction. Applying a first-order apex subtraction independently at each vertex is not a proof of such a joint extension.

Neither the free Krein formula nor a mass gap supplies the missing joint normalization, stress/current identities, or complete massless convergence theorem.

Integral reduction and the Bernoulli non-transfer

Sector D genuinely reduces to one-dimensional integrals because its two defect vertices are longitudinally translation invariant and only the neutral zero angular mode reaches the worldsheet. Equation diagonalizes their common longitudinal momentum.

Sector B does not have this property: the compact switch couples different longitudinal momenta. Even without the switch, a two-loop quartic graph generally has more than one independent momentum. A finite-rank transverse resolvent correction can simplify some factors, but it does not remove their convolution.

Similarly, the massless Bernoulli jets of I-B-G-S-v2 are finite local jets of particular scale-invariant kernels [21]. They are not a formula for a massive interacting graph. The dimensionless variables mr, extension scales, renormalization scales, and switching profiles remain. The explicit logarithm in illustrates one such surviving scale dependence.

The diagonal product-extension class of I-B-VP-3 also excludes angularly mixing boundary matrices. A theorem

for a generic non-diagonal H would enlarge that parent class and needs its own state, boundary, and graph estimates. No such enlargement is hidden in the notation here.

Reliability, exact status, and the release theorem

Evidence discipline

The proof and reporting classes are:

\texttt{ANALYTIC} A conclusion proved by a displayed argument on its stated domain, or an explicitly identified analytical input.

\texttt{SUPPLIED} A source statement or historical computation attributed to the supplied manuscript and not independently rerun.

\texttt{UNCERTIFIED} A numerical approximation without the complete justified input, truncation, quadrature, and roundoff error bounds needed for a certificate. This follows the integrity discipline of XVI-O1-ST [23].

No numerical quadrature, symbolic program, compiler run, or proof assistant was executed for this draft. The formulas in Theorems , , and are mathematical derivations, not descriptions of runs.

A future numerical comparison must retain:

extension and mass regime; exact input representations;
coupling order, graph, switch, and subtraction prescription;
raw paired values before rounding; cutoff and tail estimates;
precision, quadrature method, residuals, and failures.

A tolerance requested from an integrator is not a proved quadrature error. Near a bound threshold, uncertainty in $m^2 - \kappa_*^2$ is an independent input error.

Proposition (Finite-test coverage remains class relative). *Let a proved finite-dimensional space Z contain every admissible residual for a declared identity. A finite probe map $P : Z \to \mathbb{R}^N$ certifies that identity from vanishing probes exactly when* $\ker P = \{0\}$. *Without the containment and injectivity hypotheses, finite residual tests need not prove a distributional identity.*

Proof. The assertion within Z is the definition of injectivity. Without that property, every nonzero element of $\ker P$ is invisible to all the probes. Without proved membership in Z, the argument does not apply.

This is the supplied Oai282-A coverage distinction [24]. No continuum identity in this paper is inferred from a finite sample.

Plan-to-result audit

toprule Proposed claim Corrected result

midrule endfirsthead toprule Proposed claim Corrected result

midrule endhead bottomrule endfoot

A fixed stabilized bound-channel extension can then reach $m = 0$ above threshold False for $\kappa_* > 0$. The fixed-extension mass path stops before zero.

Order-λ^2 two-point correction is one-loop The ordinary two-vertex quartic two-point graphs have two loops. The defect Born sector is a different expansion.

The mass gap removes the second-order obstruction It controls a spectral infrared regime, not the local squared/cubed propagator extensions.

Free Hadamard subtraction already renormalizes the graph New relative-vertex collision extensions are required. They are explicitly fixed in Sector B.

Independent first-order apex subtractions suffice Joint apex/collision intersections require a separate theorem. Only the repeated quadratic defect primitive is completed here.

Every correction becomes finitely many one-dimensional integrals Proved for Sector D, not for a generic switched two-loop graph.

All stress/current Ward identities transfer from the free theory Proved only for the local bisolution sectors described here. The full interacting stress requires interaction and switch terms and compatible normalization.

All residual freedom belongs to c, τ, w False for graph normalizations. Proposition gives a regular bulk witness.

Free compact convergence proves the interacting massless limit Graph-specific estimates prove the two restricted limits and . The complete unswitched apex limit remains open.

A perturbative massless coefficient is an exact stable vacuum Not inferred. Positivity and resummed domain selection are additional obligations.

Theorem (I-B-VP-3-Ex1 mathematical release). *For the prescribed neutral Friedrichs cone and the normalizations fixed in this paper:*

1. *the relative-worldsheet primitive has the explicit renormalized Fourier transform and cutoff limit ;*

2. *the repeated quadratic defect insertion through second Born order has covariance , complete regular stress , and local scalar Ward identities;*

3. *its massless coefficients are –, including the conformal stress , with the quantitative limit ;*

4. *with $g = \lambda c/2$, its second Born term is the specified order-$\lambda^2 c^2$ contact sector;*

5. *for compact regular switched vertices separated from observations, the complete connected real-scalar two-vertex numerical coefficient is –, in the Taylor-extension prescription ;*

6. *that localized coefficient has the massless limit ;*

7. *its reflection-compatible local continuation has the exterior tensor identities ;*

8. *the fixed bound-channel massless path and the three-parameter graph-counterterm transfer asserted in the proposed plan fail for the reasons proved in Propositions and . The full unswitched second-order quartic apex theory, its complete stress and charged-current coefficients, and a positive interacting state are not released.*

Proof. Items (i)–(iv) are Theorems , , , , and , with . Items (v)–(vi) are Theorems and . Item (vii) is Proposition . Item (viii) is established by the cited propositions. The final sentence identifies constructions not provided by those results.

Remaining work and unchanged programme boundaries

Same-target continuation requirements

A continuation claiming the original full target must supply:

1. a precise real or $U(1)$-invariant complex quartic interaction, with its coupling and Wick normalization;

2. the complete order-λ^2 graph and counterterm inventory, not a single selected graph;

3. compatible extensions at bulk, apex, observation, and intersecting collision loci;

4. a specified interacting-state or real-time prescription;

5. full stress/current normalization, including switch terms and the interaction stress;

6. a graph-specific massless majorant or a proved asymptotic expansion, in a mass-admissible extension family;

7. a separate adiabatic or unswitched limit if it is wanted;

8. a complete statement of the permitted finite normalizations and their physical calibration. A necessary condition or an explicit sector calculation does not replace any missing item.

The original threshold theorem, physical core selection, general non-diagonal contacts, and the Dirac comparison remain in their separately stated scopes [17, 19, 20]. The spin-2 form-domain, core, and full Bel–Robinson questions of I-B-G-S-v2 are untouched [21].

Characteristic and interaction non-transfer

The supplied historical XIV-H record stops at H2. The later XIV-I record constructs its named I2 internal amplitude but not the full I3 joint characteristic map. XIV-J retains the unresolved J2 full-map obligation [26]. Those version-qualified statements are not collapsed into a claim that I2 was never constructed.

This paper does not construct their:

full characteristic map, joint section, complete remainder,
uniform transverse estimates, permitted-contact gluing,
output wavefront theorem, or quantum-coefficient admission.

Its conical scalar insertions are different objects. The VII-H free response and VII-I finite-width free algebra also retain their distinct domains [25].

Standing records

No version-specific external closure status is updated. For the present cone programme:

$$\mathrm{O.1} = \mathtt{texttt}\,\textit{NOT_CLOSED}, \qquad \mathtt{texttt}\,\textit{bulk_symmetry_restored} = \mathtt{texttt}\,\textit{false}.$$

XVI-N and Route B are not discharged. XV-B is not opened as a downstream quantum-admission exit. Problems 1, 2, and 8 remain unfinished; the other supplied problem numbers are not promoted by this release. The historical H2 and unresolved I3/J2 obligations are not discharged.

No Einstein dynamics, Maxwell or Einstein backreaction, spacetime emergence, Page curve, de Sitter state, cosmological bounce, or ultraviolet completion is constructed. OpenAI catalogue entries remain external records, not premises.

Conclusion. Mass can stabilize a free spatial extension without renormalizing a second-order interaction. The completed defect sector makes this distinction quantitative: its new local subtraction is explicit, its regular observables are computable, and its Friedrichs massless coefficients have a controlled limit. The localized two-vertex construction similarly shows what can be proved once supports and collision normalizations are fixed.

Neither construction justifies the original all-extension, unswitched, one-loop, three-contact claim. The resulting companion is a proof-level advance for specified sectors and a precise correction of that claim, not a declaration that the full target has been completed.

appendix

Independent algebraic checks of the defect stress

The massless second-order scalar coefficient is

$$F_{2,0} = C_{\alpha} r^{-2} \mathscr{L}, \qquad \mathscr{L}' = r^{-1}.$$

Write

$$T^{t}{}_{t,2,0} = T^{z}{}_{z,2,0} = C_{\alpha} r^{-4}(a\mathscr{L} + a_0),$$

$$T^{r}{}_{r,2,0} = C_{\alpha} r^{-4}(b\mathscr{L} + b_0), \qquad T^{\varphi}{}_{\varphi,2,0} = C_{\alpha} r^{-4}(c\mathscr{L} + c_0).$$

The coefficients in satisfy

$$a = 4\xi - \frac{2}{3}, \quad b = \frac{1}{3} - 2\xi, \quad c = 6\xi - 1,$$

$$a_0 = -4\xi + \frac{11}{18}, \quad b_0 = \xi - \frac{2}{9}, \quad c_0 = 1 - 5\xi.$$

Radial conservation reduces to two finite equations:

$$-3b - c = 0, \qquad b - 3b_0 - c_0 = 0.$$

Both hold identically. The trace has logarithmic coefficient

$$2a + b + c = 12\xi - 2$$

and constant coefficient

$$2a_0 + b_0 + c_0 = 2 - 12\xi.$$

Therefore

$$T^{\mu}{}_{\mu,2,0} = C_{\alpha} r^{-4}(12\xi - 2)(\mathscr{L} - 1) = 3\left(\xi - \frac{1}{6}\right)\mathscr{L}_r F_{2,0}.$$

This checks the conservation and trace coefficients without any symbolic-program execution.

At $\xi = 1/6$,

$$a = b = c = 0, \qquad a_0 = b_0 = -\frac{1}{18}, \qquad c_0 = \frac{1}{6},$$

which verifies the normalization and sign in . The energy density of this isolated real-scalar coefficient is

$-g^{2}T^{t}{}_{t,2,0} > 0$. This is not the energy density of the uncomputed full quartic correction and is not an Einstein

source.

Declarations

Source and authorship. The author name follows the supplied manuscript series. The complete supplied sources I-B-VP-3-v2.tex, I-B-VP-1-v4.tex, I-B-VP-1-Ex1-v2.tex, and I-B-VP-1-Ex2-v2.tex are distinguished from companion excerpts and online introductory records. Citation does not assert an independent audit of an entire corpus.

Analytical inputs. The Friedrichs heat kernel and massive local reference prescription are stated explicitly. The new relative-worldsheet subtraction, Born coefficients, massless stress coefficients, collision Taylor extensions, Wick-pairing counts, and graph-specific mass estimates are calculated in this paper. No unpublished interacting theorem is invented.

Literature and novelty. General self-interacting conical calculations, configuration-space extensions, codimension-two coupling flows, and smooth-background interacting stress normalization are prior subjects. The claimed contribution is the present normalized second-order sector package and the explicit logical correction to the proposed mass-stabilization plan. The subtraction-scale cancellation in the conformal regular stress is a result for the stated isolated defect sector, not a priority claim for all defect vacuum polarization.

State and observable scope. The free spectral state is specified. Finite perturbative numerical kernels are not automatically positive exact covariances. The local continuations are not a construction of a global in-in state or of an interacting Hamiltonian. All ordinary displayed one-point functions have regular external endpoints.

Operational and gravitational scope. The finite Enter instrument reports installed words and their actual residuals. It neither evaluates unknown continuum integrals nor selects the parameters. No continuum acquisition map, Einstein equation, or backreaction model is supplied.

Execution. No successful pdfLaTeX run, numerical quadrature, symbolic computation, proof-assistant verification, independent rebuild, registry installation, independent referee approval, or journal acceptance is

asserted.

Format. This is one pdfLaTeX source with an internal bibliography and no external figure, code, data, or bibliography-file dependency. Two runs are intended for the contents and cross-references. Submission requires the author's approval of the mathematical scope, attribution, and final source.

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