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I-B-VP-3: Massive ground states, vacuum polarization, and the bound-channel threshold

An extension-resolved construction on the prescribed regular conical Enter spacetime

Ngo Cheung

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Abstract

We construct massive scalar spectral ground states and their renormalized Wick square, mixed stress tensor, and charged azimuthal current on the regular conical product prescribed in I-B. The construction covers every positive-mass regime of the rotation-invariant, longitudinally local product-extension class defined in this paper, including non-Friedrichs extensions with negative transverse bound states.

The decisive distinction is between spectral nonnegativity and infrared admissibility. If $-\kappa_*^2$ is the lowest transverse energy, the ordinary compact-test spectral covariance exists for $m > \kappa_*$. At $m = \kappa_* > 0$, it fails precisely on tests with a nonzero zero-frequency projection onto a critical bound channel. We identify the finite-codimension restricted test space on which the threshold covariance remains meaningful and prove the universal approach-to-threshold law

$$H_{W,m} = \frac{\log(\mu/\sqrt{m^2 - \kappa_*^2})}{2\pi} \Pi_* + O_{C^\infty}(1)$$

near every regular diagonal point. Here Π_* is the critical transverse projector kernel and $\mu > 0$ is a reference scale. The resulting observable divergence is a regular-region, extension-dependent infrared effect, not an apex contact or a local ultraviolet counterterm.

For the Friedrichs family, convergent proper-time formulas evaluate all three observables for every cone angle.

For the remaining extensions, explicit Bessel resolvent coefficients reduce their observable corrections to absolutely convergent one-dimensional integrals. The complex first jet fixes the massive current, and finite-character quotient formulas independently check its sign and normalization. Conservation, the massive trace identity, curvature-coupling improvement, and compact-regular massless limits are proved.

The contribution is this simultaneous extension-resolved state-and-observable package and its exact threshold diagnosis, not a priority claim for massive cosmic-string polarization. A mass can stabilize a negative transverse channel, but critical nonnegativity does not produce an unrestricted vacuum. The cone, extension, state prescription, and normalization remain inputs. No apex quantum algebra, backreaction, interacting product, continuum acquisition by Enter readout, or Einstein dynamics is constructed.

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Book expansion: mass regimes

toprule Regime Statement class

midrule $m > \kappa_*$ Gapped construction as labelled in the state theorem

$m = \kappa_*$ Threshold; restricted test algebra

$m < \kappa_*$ Below threshold; hypotheses stated beside the theorem

bottomrule Keep the product-extension assumption and mass-dependent renormalization convention adjacent to the state theorem. The threshold restricted test algebra is *not* an unrestricted local field theory.

Purpose, interpretation, and corrections to the plan

The purpose is to complete the massive-state and regular-observable problem identified as V3 in the supplied successor plan. Historical companion ledgers distribute the massive and non-Friedrichs questions among more than one item; the identifier V3 here names this paper's stated target, not a silent renumbering of those records.

I-B prescribes the conical host and a separate finite operational interface [11]. I-B-G supplies the scalar self-adjoint-extension and propagation framework [12]. I-B-VP constructs massless spectral states and regular polarization observables [13]; I-B-VP-0 evaluates the charged massless current [14]. I-B-VP-1 concerns prescribed apex finite parts [15], whereas I-B-VP-2 concerns the ordinary global massless wavefront relation on the regular host [16].

Massive cosmic-string polarization and induced currents are established subjects. In particular, massive charged-current formulas and their distinction from finite-core contributions are available in [3]. The role of

boundary conditions at an idealized string is classical [1], and inverse-square radial extensions have an explicit Bessel spectral theory [2]. We do not claim that massive polarization or stabilization by a mass originates in the Enter framework.

The mathematical class is fixed before the calculation

An extension-dependent theorem needs an actual domain class. We analyze all separated, rotation-invariant apex conditions whose parameters are independent of z and of longitudinal momentum. Equivalently, a single transverse self-adjoint operator is tensored with the ordinary longitudinal Laplacian. This is the product class defined explicitly in Section .

Momentum-dependent apex conditions, angularly mixing conditions, finite-core models, and arbitrary nonlocal boundary operators are not included. Without additional hypotheses, they cannot be assigned the same state, symmetry, or azimuthal-observable theorem.

The supplied I-B-VP-v4 excerpts specify the renormalization and Ward conventions but do not include a complete Krein theorem or its estimates. We therefore derive the resolvent coefficients and all estimates used below. No unavailable companion estimate is treated as a proved premise.

Corrections needed for a valid massive theorem

1. Opening versus deficit. The parameter $\alpha \in (0, 1)$ is the opening ratio. The deficit angle is $2\pi(1 - \alpha)$.

2. Mass stabilization does not require a nonnegative transverse operator. The relevant condition is $m^2 - \kappa_*^2 > 0$, not nonnegativity of the transverse extension itself. There are also non-Friedrichs extensions with no negative transverse eigenvalue; every $m > 0$ is gapped for those extensions.

3. The extension correction is not globally smooth at arbitrary Lorentzian pairs. It is jointly analytic on the regular Euclidean product and analytic in a sufficiently small Lorentzian neighbourhood of each regular diagonal point. Additional global reflected or diffracted singularities are not excluded by this local statement.

4. The massive subtraction and ambiguity both change. The massless direct kernel cannot be subtracted

unchanged. The Wick-square freedom $a_{\mathrm{ren}} m^2$ and the flat stress freedom $b_{\mathrm{ren}} m^4 g_{\mu\nu}$ both survive before normalization.

5. Flat limits depend on the extension. The Friedrichs observables tend to zero when the geometry and holonomy become trivial. A retained non-Friedrichs apex condition can remain a point defect in the flat limit and need not give zero polarization.

6. Threshold channels are visible on regular tests. Their transverse bound functions are nonzero for $r > 0$. Only tests annihilating the relevant channel fail to distinguish changes confined to it. Regular support by itself does not make a threshold channel invisible.

7. Mass shells and ordinary wavefront sets are different. Mass changes the dispersion relation, but the principal symbol of the Klein–Gordon operator remains null. This paper makes no assertion that the ordinary wavefront set becomes a massive mass shell.

Enter philosophy and spacetime

The Philosophy of Enter distinguishes destructive readout from quantum recovery, a classical record from a decoder, and an analogy from an identification [10]. In the spacetime model I-B, a finite clock interface is placed on an independently prescribed continuum geometry [11]. Its record does not supply an apex domain, a scalar vacuum, or a continuum covariance.

These distinctions are interpretive boundaries, not hypotheses needed in the Bessel calculation. The field, mass, extension, state prescription, and renormalization convention below are declared mathematical inputs. Conservation does not derive the cone or impose Einstein dynamics.

Host, radial extensions, and the exact spectral ledger

Geometry and normalization

Fix

$$0 < \alpha < 1, \qquad 0 \le \beta < 1, \qquad m > 0.$$

The regular spacetime is

$$M = \mathbb{R}_t \times (0,\infty)_r \times (\mathbb{R}/2\pi\mathbb{Z})_\varphi \times \mathbb{R}_z, \qquad g = -\mathrm{d}t^2 + \mathrm{d}r^2 + \alpha^2 r^2 \mathrm{d}\varphi^2 + \mathrm{d}z^2.$$

Its completed defect is the worldsheet $\mathscr{W} = \mathbb{R}_t \times \{o\} \times \mathbb{R}_z$, which is not part of M.

In a lifted parallel frame,

$$\Phi(\varphi + 2\pi) = e^{2\pi i\beta}\Phi(\varphi).$$

The regular bundle curvature and Riemann tensor vanish. The equation is

$$(\mathrm{Box}_g - m^2)\Phi = 0.$$

The coupling ξ enters the stress improvement, not this regular equation or the chosen apex domain. No distributional ξR interaction is modeled.

Put

$$\ell_n = n + \beta, \qquad \nu_n = \frac{|\ell_n|}{\alpha}, \qquad e_n(\varphi) = \frac{e^{i\ell_n\varphi}}{\sqrt{2\pi\alpha}}.$$

The transverse measure is

$$\mathrm{d}V_\perp = \alpha r\, \mathrm{d}r\, \mathrm{d}\varphi.$$

The radial expression on $L^2((0,\infty), r\,\mathrm{d}r)$ is

$$h_\nu = -\partial_r^2 - r^{-1}\partial_r + \nu^2 r^{-2}.$$

Definition (Product extension class). For every mode with $\nu_n \ge 1$, take the unique self-adjoint realization at the apex.

For $0 < \nu_n < 1$, use the boundary coefficients

$$u(r) = a\, r^{-\nu_n} + b\, r^{\nu_n} + o(r^{\nu_n})$$

in the usual maximal-domain boundary-value sense. Choose either the Friedrichs condition $a = 0$, or

$$b = \lambda_n a, \qquad \lambda_n \in \mathbb{R}.$$

For $\nu_n = 0$, choose either the Friedrichs condition of vanishing logarithmic coefficient, or

$$u(r) = a\log(r/L_n) + o(1), \qquad L_n > 0.$$

All extension parameters are fixed, independent of t, z, m, and longitudinal momentum.

Let H_W be the transverse direct sum of these radial operators, including the angular factors e_n. Define

$$A_W = H_W \otimes I + I \otimes (-\partial_z^2), \qquad B_{W,m} = A_W + m^2.$$

The entirely Friedrichs choice is denoted $W = \mathrm{F}$.

There are only finitely many non-Friedrichs slots:

$$\mathscr{I}_\beta = \{n \in \mathbb{Z} : |n + \beta| < \alpha\}.$$

For $0 < \alpha < 1$, this set has at most two elements. The boundary parameters in have dimensions length$^{-2\nu_n}$. Changing their normalization changes their numerical values, so the displayed boundary convention is part of the theorem.

For a real neutral scalar take $\beta = 0$, with the real domain. Charged conjugation statements require the extension family to be conjugated along with the bundle.

Transverse bound scales

Define

$$d_\nu = 2^{2\nu}\frac{\Gamma(1+\nu)}{\Gamma(1-\nu)}, \qquad 0 < \nu < 1.$$

Proposition (Transverse negative spectrum). *A power-law slot has one negative eigenvalue precisely when*

$\lambda_n < 0$, *namely*

$$-\kappa_n^2, \qquad \kappa_n^{2\nu_n} = -\lambda_n d_{\nu_n}.$$

A logarithmic slot has one negative eigenvalue with

$$\kappa_n = \frac{2e^{-\gamma_E}}{L_n}.$$

There are no other negative eigenvalues. The remaining transverse spectrum is $[0,\infty)$*, and there is no transverse* L^2 *zero eigenfunction.*

Proof. A decaying solution at energy $-q^2$, $q > 0$, is $K_\nu(qr)$. Its coefficients for $0 < \nu < 1$ are

$$a_K(q) = 2^{\nu-1}\Gamma(\nu)q^{-\nu}, \qquad b_K(q) = 2^{-\nu-1}\Gamma(-\nu)q^{\nu}.$$

Thus $b_K/a_K = \lambda$ is possible precisely for $\lambda < 0$, and gives . For $\nu = 0$,

$$K_0(qr) = -\log r + \log(2/q) - \gamma_E + o(1),$$

and gives . The decaying solution is unique, and each scalar boundary equation has exactly the root stated.

At positive energy, the Bessel solutions oscillate at infinity and are not square integrable with measure $r\,\mathrm{d}r$. At zero energy the power and logarithmic solutions cannot be square integrable at both ends. The continuous spectral interval follows from the Bessel resolvent boundary values, or the generalized Hankel diagonalization for these real self-adjoint domains [2]. The small-argument expansions used above are standard Bessel identities [7].

Let $\mathscr{B}$ be the finite collection of bound channels. Set

$$\kappa_* = \begin{cases} \max_{j\in\mathscr{B}} \kappa_j, & \mathscr{B} \neq \varnothing, \\ 0, & \mathscr{B} = \varnothing. \end{cases}$$

A normalized bound function is

$$\chi_j(r,\varphi) = e_{n_j}(\varphi)\, N_j K_{\nu_j}(\kappa_j r),$$

where

$$N_j^2 = \begin{cases} \dfrac{2\kappa_j^2 \sin(\pi\nu_j)}{\pi\nu_j}, & 0 < \nu_j < 1, \\ 2\kappa_j^2, & \nu_j = 0. \end{cases}$$

Indeed,

$$\int_0^\infty r K_\nu(\kappa r)^2 \, \mathrm{d}r = \frac{\pi\nu}{2\kappa^2 \sin(\pi\nu)}$$

with its continuous $\nu = 0$ value $1/(2\kappa^2)$. This identity also follows by integrating the radial Lagrange identity and using the two endpoint expansions.

Theorem (Mass stabilization and absence of product point spectrum). *For the class of Definition ,*

$$\operatorname{spec}(B_{W,m}) = [m^2 - \kappa_*^2, \infty), \qquad B_{W,m} \geq 0 \iff m \geq \kappa_*.$$

The full spatial operator has no L^2 point spectrum. If $m > \kappa_*$, its gap is

$$\delta_m = m^2 - \kappa_*^2 > 0.$$

Proof. Fourier transform in z gives the direct integral

$$B_{W,m} \simeq \int_{\mathbb{R}}^{\oplus} (H_W + m^2 + p^2) \, \mathrm{d}p.$$

The continuous transverse spectrum gives $[m^2, \infty)$. A bound channel contributes the band

$$\{m^2 - \kappa_j^2 + p^2 : p \in \mathbb{R}\} = [m^2 - \kappa_j^2, \infty).$$

Their union proves .

For any putative eigenvalue E, use a spectral representation of H_W. For each transverse spectral value λ, the equation $\lambda + m^2 + p^2 = E$ holds on a set of longitudinal Lebesgue measure zero. Fubini's theorem therefore makes the corresponding full spectral projection zero.

This proves directly the positivity ledger associated with I-B-G thm:positivity [12]. An attractive transverse extension is not excluded merely because H_W has a negative eigenvalue.

Gapped states and the exact threshold restriction

The ordinary gapped spectral state

Theorem (Ground state above threshold). *If $m > \kappa_*$, then*

$$W_{W,m}(T; x, x') = \left[\frac{e^{-\mathrm{i}T\sqrt{B_{W,m}}}}{2\sqrt{B_{W,m}}}\right](x, x'), \qquad T = t - t',$$

defines a continuous positive covariance on all compact smooth regular spacetime tests. It gives the zero-mean quasifree spectral Fock ground state.

Its commutator has the extension-selected normalization

$$[\Phi(X), \Phi^\dagger(X')] = -\mathrm{i}E_{W,m}(X, X'), \qquad E_{W,m}(T) = \frac{\sin(T\sqrt{B_{W,m}})}{\sqrt{B_{W,m}}}.$$

For a charged field the reversed ordering is taken in the conjugate sector, not by replacing the complex kernel with its pointwise real part.

Proof. The spectral gap gives the explicit estimate

$$\left\|(2\sqrt{B_{W,m}})^{-1/2} v\right\| \le \frac{\|v\|}{\sqrt{2}\,\delta_m^{1/4}}.$$

Consequently

$$\mathscr{K}_{W,m} f = \int_{\mathbb{R}} e^{\mathrm{i}t\sqrt{B_{W,m}}}(2\sqrt{B_{W,m}})^{-1/2} f(t)\,\mathrm{d}t$$

exists, and

$$\|\mathscr{K}_{W,m} f\| \le \frac{1}{\sqrt{2}\,\delta_m^{1/4}} \int \mathrm{lVert f}(t)\|\,\mathrm{d}t.$$

This proves continuity in every fixed-support test topology. The Gram form

$$W_{W,m}(f, g) = \langle \mathscr{K}_{W,m} f, \mathscr{K}_{W,m} g\rangle$$

proves positivity. The standard particle/antiparticle Fock construction has nonnegative one-particle Hamiltonian

$\sqrt{B_{W,m}}$ [9]. Time integration by parts proves the field equation. The difference of the two time orientations is

$$\frac{e^{-\mathrm{i}T\sqrt{B}} - e^{\mathrm{i}T\sqrt{B}}}{2\sqrt{B}} = -\mathrm{i}\frac{\sin(T\sqrt{B})}{\sqrt{B}},$$

which gives .

This estimate proves compact-test finiteness; nonnegativity alone was not substituted for it.

Critical moments

Assume now $m = \kappa_* > 0$, and let

$$\mathscr{C}_* = \{j \in \mathscr{B} : \kappa_j = \kappa_*\}.$$

For a spatial test v, define

$$v_j(z) = \int \overline{\chi_j(\zeta)} v(\zeta, z)\, \mathrm{d}V_\perp(\zeta), \qquad \hat{v}_j(p) = \frac{1}{\sqrt{2\pi}} \int e^{-\mathrm{i}pz} v_j(z)\, \mathrm{d}z.$$

For a spacetime test f, define $f_j(t,z)$ similarly and put

$$M_j(f) = \int_{\mathbb{R}^2} f_j(t,z)\, \mathrm{d}t\, \mathrm{d}z.$$

Theorem (Sharp threshold obstruction and restricted domain). *At $m = \kappa_* > 0$, the equal-time spectral quadratic form is finite on a compact smooth spatial test v if and only if*

$$\hat{v}_j(0) = 0 \qquad (j \in \mathscr{C}_*).$$

The spacetime spectral form is finite on a compact smooth test f if and only if

$$M_j(f) = 0 \qquad (j \in \mathscr{C}_*).$$

In particular, the ordinary spectral covariance does not exist on all compact regular tests.

On

$$\mathscr{D}_* = \{f \in C_c^\infty(M; L_\beta) : M_j(f) = 0 \text{for every} j \in \mathscr{C}_*\},$$

the spectral form is continuous and positive and defines a quasifree ground state on the correspondingly restricted CCR test algebra. This is not the unrestricted field state.

Proof. On a critical bound channel,

$$B_{W,\kappa_*} = -\partial_z^2,$$

so its equal-time contribution is

$$\int_{\mathbb{R}} \frac{|\hat{v}_j(p)|^2}{2|p|}\,\mathrm{d}p.$$

If $\hat{v}_j(0) \neq 0$, continuity gives logarithmic divergence. If it vanishes, smooth compact support gives

$$|\hat{v}_j(p)| \leq C|p|$$

near zero, and converges. The other transverse channels have a strictly positive gap, because $\kappa_* > 0$ and only finitely many bound scales occur.

For a spacetime test the critical contribution is

$$\int_{\mathbb{R}} \frac{1}{2|p|}\left|\frac{1}{\sqrt{2\pi}}\int e^{\mathrm{i}|p|t\,-\,\mathrm{i}pz} f_j(t,z)\,\mathrm{d}t\,\mathrm{d}z\right|^2 \mathrm{d}p.$$

Its numerator tends to $|M_j(f)|^2/(2\pi)$ at $p = 0$. If the moment vanishes, Taylor's theorem gives an $O(|p|^2)$ squared numerator. At high momentum, compact smooth spacetime support gives rapid decay. These estimates also prove continuity on $\mathscr{D}_*$ with fixed-support seminorms.

The form is positive. The same spectral construction as above gives the restricted quasifree state. The moments vanish on Klein–Gordon images of compact tests, since integration in t, z removes their longitudinal derivatives and $(H_W + \kappa_*^2)\chi_j = 0$. Thus the restriction is compatible with imposing the field equation.

Finally, the failing class is nonempty on regular supports. Choose a compact transverse cutoff $\eta \geq 0$, not zero, and take a test proportional to $\eta\chi_j$, times compact longitudinal and temporal bumps with nonzero integrals. Its j

-moment is nonzero. The obstruction is not caused by testing at the apex.

The conditions are finite in number, but they are global moment conditions on the test. They do not define an unrestricted local scalar field by another name. Longitudinal derivatives of tests automatically remove the critical zero moment; transverse differentiation does not generally do so.

Below threshold

Proposition (Negative-band exclusion and projected substitutes). *If $m < \kappa_*$, the nonnegative square-root ground-state prescription is unavailable for the original product field.*

Projecting onto the nonnegative spectral part of $B_{W,m}$ and projecting away all transverse bound channels are different modifications. Neither preserves the original canonical equal-time commutator on the full test space.

Proof. Theorem gives a negative spectral band. There is no nonnegative self-adjoint square root of that operator.

In a bound channel with $\kappa_j > m$, the first projection retains longitudinal momenta

$$|p| \geq \sqrt{\kappa_j^2 - m^2},$$

whereas the second removes the entire channel, including its positive-energy high-momentum part. The two projections therefore differ. The corresponding equal-time commutator is the selected projection rather than the identity. Their kernels are nonlocal spatial modifications. Further infrared checks would be required before assigning states to every projected substitute.

Massive Friedrichs covariance and local subtraction

Heat integration

Let $K_{\mathrm{F},\beta}(s; x, x')$ be the massless spatial Friedrichs heat kernel:

$$K_{\mathrm{F},\beta}(s; x, x') = \frac{e^{-(z-z')^2/(4s)}}{\sqrt{4\pi s}} \frac{e^{-(r^2+r'^2)/(4s)}}{4\pi\alpha s}$$

$$\times \sum_{n \in \mathbb{Z}} e^{i\ell_n(\varphi-\varphi')} I_{\nu_n}\left(\frac{rr'}{2s}\right).$$

This is the Bessel heat kernel used in I-B-G [12, thm:closed]; multiplication by $e^{-m^2 s}$ gives the massive heat kernel. We use that identity without recomputing the retarded massive kernel.

The Euclidean covariance is

$$G^E_{\mathrm{F},m} = \int_0^\infty \frac{e^{-m^2 s - (\tau-\tau')^2/(4s)}}{\sqrt{4\pi s}} K_{\mathrm{F},\beta}(s; x, x')\, \mathrm{d}s.$$

It is the kernel of

$$\frac{e^{-|\tau-\tau'|\sqrt{B_{\mathrm{F},m}}}}{2\sqrt{B_{\mathrm{F},m}}}.$$

In a sufficiently small parallel developing chart, write

$$\ell_E^2 = r^2 + r'\text{^}2 - 2rr'\cos(\alpha(\varphi-\varphi')) + (\tau-\tau')^2 + (z-z')^2.$$

The direct massive fundamental kernel is

$$P^E_m(\ell_E) = \frac{mK_1(m\ell_E)}{4\pi^2 \ell_E}.$$

At $m = 0$, its continuous limit is $1/(4\pi^2\ell_E^2)$.

Lemma (Interior heat localization). *On a fixed compact regular developing product neighbourhood of the diagonal, the difference between and the direct flat spatial heat kernel is $O(s^N)$ as* sdownarrow 0, *for every N, after any prescribed finite number of endpoint derivatives. The estimates can be chosen uniformly when α, β range in a sufficiently small compact parameter neighbourhood with the same regular charts.*

Proof. Choose nested regular charts and a cutoff equal to one on the smaller chart. The local differential expression in a parallel developing frame is exactly the flat Laplacian. The cutoff flat heat kernel has the correct initial delta distribution there; its error is supported where the cutoff varies, a positive distance from the smaller chart.

The heat semigroup correction is obtained by Duhamel's formula. The Friedrichs form is the closure of the local

covariant Dirichlet energy. The usual weighted energy estimate for this form gives off-diagonal Gaussian decay between the separated supports. Applying local interior parabolic estimates transfers the same decay, with finite inverse powers of s, to any fixed endpoint derivatives. Such bounds imply $O(s^{N})$ for every N. The separation and local ellipticity constants have positive uniform bounds in the stated parameter neighbourhood.

Equivalently, the image–diffraction heat identity of I-B-G yields this localized estimate when treated as a total kernel. No pointwise choice at a changing image boundary is needed.

Theorem (Massive local Hadamard remainder). *For $m > 0$,*

$$H^{E}_{\mathrm{F},m} = G^{E}_{\mathrm{F},m} - P^{E}_{m}$$

is jointly real-analytic across the regular diagonal on each sufficiently small developing product chart. Its local Lorentzian continuation gives

$$W_{\mathrm{F},m} = P^{+}_{m} + H_{\mathrm{F},m},$$

with smooth, locally analytic $H_{\mathrm{F},m}$. The direct term P^{+}_{m} is the massive Minkowski positive-frequency kernel in that chart.

Proof. The Euclidean kernel in is a fundamental distribution for $-\Delta_{E} + m^{2}$ in each endpoint. Its large-s convergence follows from $e^{-m^{2}s}$; its local small-s subtraction is justified by Lemma . The direct kernel has exactly the same local delta source. The difference is therefore annihilated by both endpoint operators across the diagonal. Their sum is elliptic on the product chart and has analytic coefficients. Analytic elliptic regularity makes the difference jointly real-analytic [8]. Its holomorphic continuation in a small time-difference neighbourhood agrees with the spectral continuation from $e^{-a\sqrt{B}}/(2\sqrt{B})$, $\mathrm{Re}\, a > 0$. Taking $a = \varepsilon + \mathrm{i}(t - t')$ proves the Lorentzian statement.

The small-distance expansion

$$P^{E}_{m}(\ell) = \frac{1}{4\pi^{2}\ell^{2}} + \frac{m^{2}}{8\pi^{2}}\left(\log(m\ell/2) + \gamma_{E} - \frac{1}{2}\right) + O(\ell^{2}\log\ell)$$

shows explicitly why unmodified massless subtraction is insufficient.

Explicit resolvent correction for all product extensions

The radial coefficients

For a non-Friedrichs power slot define

$$c_n(q) = \frac{2\sin(\pi\nu_n)}{\pi}\frac{q^{2\nu_n}}{q^{2\nu_n} + \lambda_n d_{\nu_n}}, \qquad q > \kappa_*.$$

For a logarithmic slot define

$$c_n(q) = \frac{1}{\log(q/\kappa_n)}.$$

Friedrichs slots have $c_n = 0$.

Proposition (Radial Krein formula with fixed conventions). *Relative to $r'\,\mathrm{d}r'$, the radial resolvent at $-q^2$, $q > \kappa_*$, is*

$$(h_{\nu_n, W} + q^2)^{-1}(r, r') = I_{\nu_n}(qr_<)K_{\nu_n}(qr_>) + c_n(q)K_{\nu_n}(qr)K_{\nu_n}(qr').$$

Its only positive-q poles are the bound scales already listed.

Proof. The Friedrichs Green kernel is the first term, by the I_ν, K_ν Wronskian. Any other resolvent with the same source and decaying infinity condition differs by a multiple of $K_\nu(qr)K_\nu(qr')$.

For $0 < \nu < 1$, impose the boundary condition on

$$I_\nu(qr) + c(q)K_\nu(qr).$$

The regular coefficient of I_ν is $i_\nu(q) = (q/2)^\nu/\Gamma(1+\nu)$. Thus

$$i_\nu + c\, b_K = \lambda c\, a_K, \qquad c = \frac{i_\nu}{\lambda a_K - b_K}.$$

The gamma reflection identity reduces this to . For $\nu = 0$, the constant and logarithmic coefficients of $I_0 + cK_0$ give

$$1 + c(\log(2/q) - \gamma_E) = c\log L,$$

which is . The denominators give exactly the bound-state equations.

The formula is a particular explicit Krein resolvent identity, not an appeal to an unspecified correction map. It is consistent with the general Bessel extension analysis in [2].

Euclidean state correction

Suppose $m > \kappa_*$. Put

$$q = \sqrt{m^2 + \omega^2 + p^2}.$$

Define

$$\mathscr{C}^E_{W,m}(X, X') = \frac{1}{2\pi\alpha} \sum_{n \in \mathscr{I}_\beta} e^{i\ell_n(\varphi - \varphi')} \int_{\mathbb{R}^2} \frac{d\omega\, dp}{(2\pi)^2} e^{i\omega(\tau - \tau') + ip(z - z')}$$

$$\times c_n(q) K_{\nu_n}(qr) K_{\nu_n}(qr').$$

Zero coefficients are simply omitted.

Theorem (Above-threshold extension correction). *For every extension in Definition and every $m > \kappa_*$,*

$$G^E_{W,m} = G^E_{\mathrm{F},m} + \mathscr{C}^E_{W,m}.$$

The correction is jointly real-analytic for regular Euclidean endpoints. It continues analytically to a Lorentzian neighbourhood of every regular diagonal point. Consequently

$$H_{W,m} = H_{\mathrm{F},m} + \mathscr{C}_{W,m}$$

is a valid local massive Hadamard remainder.

No global Lorentzian smoothness of $\mathscr{C}_{W,m}$ is asserted.

Proof. Fourier transformation in (τ, z) turns the Euclidean fundamental operator into $H_W + m^2 + \omega^2 + p^2$.

Proposition gives and .

On a compact regular set with $r, r' \geq r_0 > 0$, the finite collection of coefficients $c_n(q)$ is smooth for $q \geq m > \kappa_*$, bounded at infinity, and has polynomially bounded derivatives. The standard large-q Bessel estimate gives

$$K_\nu(qr)K_\nu(qr') = O(q^{-1}e^{-q(r+r')}).$$

Any fixed endpoint derivative adds only polynomial factors. Therefore and all its derivatives converge absolutely.

The same estimates hold in sufficiently small complex neighbourhoods of all real endpoint coordinates. In particular, complex longitudinal differences with imaginary part of Euclidean norm less than $r + r'$ are allowed after reducing the neighbourhood. Bessel analyticity and dominated holomorphic integration prove joint real analyticity and local Lorentzian continuation [7]. The spectral Euclidean identity then identifies the continuation with the state of Theorem . Adding it to Theorem proves the result.

Remark (Bound-state pieces must not be isolated in an ultraviolet proof). A spectral decomposition into a transverse bound channel and transverse continuum is useful for infrared analysis. The two pieces need not separately have the local four-dimensional Hadamard singularity. The exponentially convergent resolvent *difference* in proves local smoothness after their proper assembly.

Renormalization, stress normalization, and Ward identities

Reference prescription and finite freedom

On regular developing charts subtract the full massive Minkowski kernel P_m^+, including its bundle parallel factor. This is a locally defined reference prescription: different developing charts are related by local isometries and parallel gauge transformations.

For one real neutral scalar set $n_{\mathrm{f}} = 1$; for one charged complex scalar set $n_{\mathrm{f}} = 2$. Define

$$F = [H_{W,m}], \qquad h = n_{\mathrm{f}} \operatorname{Re} H_{W,m},$$

so the field square entering the stress trace is

$$F_h = [h] = n_f F.$$

For the complex field, F means $\langle \Phi^\dagger \Phi \rangle_{\mathrm{ren}}$.

Assumption (Regular renormalization class). Renormalization is local and gauge covariant, state independent, and has the standard engineering dimensions. No Wilson loop, radial vector, angular Killing field, extension parameter, or apex-supported distribution is admitted as an extra local subtraction datum.

For the explicit formulas below, require both the Wick square and stress to vanish in the ordinary massive Minkowski vacuum with trivial connection, at the fixed mass under consideration. The current has the fixed unit-charge Noether normalization.

Before imposing that normalization, the allowed flat-background freedoms include

$$F \operatorname{longmapsto} F + a_{\mathrm{ren}} m^2, \qquad T^\mu{}_\nu \operatorname{longmapsto} T^\mu{}_\nu + b_{\mathrm{ren}} m^4 \delta^\mu_\nu.$$

The usual curvature and gauge-curvature terms vanish on the regular host. The current ambiguity proportional to $\nabla^\nu F_{\nu\mu}$ also vanishes [4, 5, 6]. Thus the massive Wick freedom is not the only surviving mass-dependent freedom.

Remark (Mass normalization versus smoothness at zero mass). Subtracting the full massive Minkowski covariance is a mass-dependent normalization. Relative to a fixed Hadamard length scale, the corresponding flat reference contains $m^2 \log m$ and stress terms of order $m^4 \log m$.

We impose the prescription for $m > 0$ and prove a continuous massless limit of the conical observables. We do not claim that this choice simultaneously gives analytic or C^∞ dependence on m^2 through $m = 0$ in a fixed-scale renormalization family. Keeping that stronger requirement instead would require retaining its explicit Minkowski reference terms.

Relative to the full massive Minkowski reference, Assumption fixes $a_{\mathrm{ren}} = b_{\mathrm{ren}} = 0$. If an alternative pair in is retained, the trace identity below acquires the constant $(4b_{\mathrm{ren}} + n_f a_{\mathrm{ren}})m^4$.

Local formulas

Put

$$A_{\mu\nu} = [\nabla_\mu \nabla_\nu h], \qquad K = A^\mu{}_\mu.$$

The stress can be written

$$T_{\mu\nu} = A_{\mu\nu} - \frac{1}{2} g_{\mu\nu}(K + m^2 F_h) + \xi(g_{\mu\nu}\mathrm{Box}_g - \nabla_\mu \nabla_\nu) F_h.$$

Primed indices are parallel transported before coincidence. On the flat regular host this is the usual scalar stress for the stated action normalization.

For the charged field define

$$\mathscr{J}_\mu = \mathrm{i}[(D_\mu - D_\mu^*) H_{W,m}], \qquad J_\mu = \mathrm{i}\{\Phi^\dagger D_\mu \Phi - (D_\mu \Phi)^\dagger \Phi\}.$$

An electric current of charge e is $e\mathscr{J}_\mu$ with the same connection convention.

Theorem (Massive Ward identities). *For every $m > \kappa_*$, the regular observables are smooth and real and obey*

$$\nabla_\mu T^\mu{}_\nu = 0, \qquad \nabla_\mu \mathscr{J}^\mu = 0,$$

$$T^\mu{}_\mu = 3\left(\xi - \frac{1}{6}\right)\mathrm{Box}_g F_h - m^2 F_h,$$

$$T_{\mu\nu}(\xi) - T_{\mu\nu}(1/6) = \left(\xi - \frac{1}{6}\right)(g_{\mu\nu}\mathrm{Box}_g - \nabla_\mu \nabla_\nu) F_h.$$

Here the extension is held fixed while ξ changes. The current is independent of ξ.

Proof. The local remainder is a smooth massive bisolution. In a parallel Cartesian chart, Synge's rule gives

$$\mathrm{Box}\, F_h = 2K + 2m^2 F_h, \qquad \partial^\mu A_{\mu\nu} = \frac{1}{4}\partial_\nu \mathrm{Box}\, F_h.$$

Consequently is equivalently

$$T_{\mu\nu} = A_{\mu\nu} - \xi \partial_\mu \partial_\nu F_h + \left(\xi - \frac{1}{4}\right) g_{\mu\nu}\mathrm{Box}\, F_h.$$

Its divergence vanishes. Taking its trace and using $K = \frac{1}{2}\mathrm{Box}\, F_h - m^2 F_h$ proves . Subtracting the same formula at $\xi = 1/6$ proves .

Hermiticity proves reality of the current. Its divergence is

$$\mathrm{i}[(\mathrm{Box}_X - \mathrm{Box}_{X'})H_{W,m}] = 0,$$

because both endpoint equations have the same mass. Covariant endpoint differentiation makes the expression gauge invariant. The equation, fixed domain, and current normalization are independent of ξ.

The trace $-m^2 F_h$ at $\xi = 1/6$ is explicit mass breaking, not a curvature anomaly. In the selected flat Minkowski-normalized prescription, no additional state-independent flat trace term remains.

Remark (The complex first jet is indispensable). Hermiticity gives

$$\mathscr{J}_\mu = -2\mathrm{Im}\,[\nabla_\mu H_{W,m}].$$

Replacing $H_{W,m}$ by its real part before taking this jet would erase the circulating current. This is the same normalization distinction as in I-B-VP-0 [14], now with a massive covariance.

In the periodic gauge, $H_{\mathrm{per}} = e^{-\mathrm{i}\beta(\varphi - \varphi')}H_{\mathrm{par}}$. The endpoint covariant derivatives and the transformed parametrix cancel that phase. Ordinary derivatives of the periodic kernel alone are not the prescription.

Evaluated regular observables

All formulas in this section are in the Minkowski-normalized prescription. They are convergent integral evaluations, not claims of an elementary polynomial closed form.

Four real covariance moments

Let

$$D(r) = F''(r) + r^{-1}F'(r) = \mathscr{L}_r F(r).$$

Define the moments of the single-copy remainder by

$$L = [\partial_\tau \partial_{\tau'} \mathrm{Re}\, H^E], \qquad R = [\partial_r \partial_{r'} \mathrm{Re}\, H^E],$$

$$V = \frac{1}{\alpha^2 r^2}[\partial_\varphi \partial_{\varphi'} \mathrm{Re}\, H^E].$$

The letter R here denotes a radial moment, not scalar curvature, which is zero on M.

Rotational and longitudinal symmetries make these functions of r. Euclidean rotational symmetry in (τ, z), followed by continuation, gives

$$[\nabla^t \nabla_{t'} \mathrm{Re}\, H] = [\nabla^z \nabla_{z'} \mathrm{Re}\, H] = L.$$

Proposition (Complete mixed stress from the moments). *In component order* (t, r, φ, z),

$$T^t{}_t = T^z{}_z = n_\mathrm{f}\Big[L + \Big(\xi - \frac{1}{4}\Big)D\Big],$$

$$T^r{}_r = n_\mathrm{f}\Big[R - \xi F'' + \Big(\xi - \frac{1}{4}\Big)D\Big],$$

$$T^\varphi{}_\varphi = n_\mathrm{f}\Big[V - \xi\frac{F'}{r} + \Big(\xi - \frac{1}{4}\Big)D\Big].$$

All off-diagonal components vanish. Moreover,

$$2L + R + V = \frac{1}{2}D - m^2 F.$$

Proof. The radial scalar has mixed Hessian entries $(0, F'', F'/r, 0)$. Equation therefore gives . A radial–angular cross derivative of an individual mode is imaginary at coincidence; its real stress part vanishes. The other mixed terms vanish by longitudinal parity and symmetry. The identity is the trace $K = \frac{1}{2}\mathrm{Box}\, F - m^2 F$ for one copy.

In particular, the stress is not reconstructed from the Wick square alone; the longitudinal moment L contains additional covariance information.

Friedrichs proper-time formulas, valid at all angles

For $x > 0$, define the already-summed functions

$$\mathscr{H}(x) = \frac{e^{-x}}{\alpha} \sum_{n \in \mathbb{Z}} I_{\nu_n}(x) - 1,$$

$$\mathscr{R}(x) = \frac{e^{-x}}{\alpha} \sum_{n \in \mathbb{Z}} \left[\left(2x + \frac{\nu_n^2}{x} \right) I_{\nu_n}(x) - 2x I'_{\nu_n}(x) \right] - 1,$$

$$\mathscr{V}(x) = \frac{e^{-x}}{\alpha} \sum_{n \in \mathbb{Z}} \frac{\nu_n^2}{x} I_{\nu_n}(x) - 1,$$

$$\mathscr{A}(x) = e^{-x} \sum_{n \in \mathbb{Z}} \ell_n I_{\nu_n}(x).$$

Each sum is absolutely convergent for fixed $x > 0$.

Theorem (Massive Friedrichs observables). *Set $x = r^2/(2s)$. Then*

$$F_{\mathrm{F},m}(r) = \frac{1}{16\pi^2} \int_0^\infty s^{-2} e^{-m^2 s} \mathscr{H}(x)\, \mathrm{d}s,$$

$$L_{\mathrm{F},m}(r) = \frac{1}{32\pi^2} \int_0^\infty s^{-3} e^{-m^2 s} \mathscr{H}(x)\, \mathrm{d}s,$$

$$R_{\mathrm{F},m}(r) = \frac{1}{32\pi^2} \int_0^\infty s^{-3} e^{-m^2 s} \mathscr{R}(x)\, \mathrm{d}s,$$

$$V_{\mathrm{F},m}(r) = \frac{1}{32\pi^2} \int_0^\infty s^{-3} e^{-m^2 s} \mathscr{V}(x)\, \mathrm{d}s,$$

$$\mathscr{J}_{\varphi,\mathrm{F},m}(r) = -\frac{1}{8\pi^2 \alpha} \int_0^\infty s^{-2} e^{-m^2 s} \mathscr{A}(x)\, \mathrm{d}s.$$

The stress is . The other current components vanish. The physical angular current is

$$\mathscr{J}_{\varphi}^{\wedge} = \frac{\mathscr{J}_{\varphi}}{\alpha r}.$$

All integrals and all fixed regular radial derivatives converge. The formulas hold for every $0 < \alpha < 1$, including image-boundary opening ratios.

Proof. At equal Euclidean times and longitudinal coordinates, the four-dimensional heat integrand in is

$$\frac{e^{-m^2 s}}{16\pi^2 \alpha s^2} e^{-(r^2 + r'^{\wedge}2)/(4s)} \sum_n e^{\mathrm{i}\ell_n(\varphi - \varphi')} I_{\nu_n}(rr'/(2s)).$$

Its direct flat subtraction gives $\mathscr{H}$. Two opposite longitudinal endpoint derivatives multiply the integrand by $1/(2s)$

, giving L.

For the radial derivatives, the Bessel equation gives, at $r' = r$,

$$\partial_r \partial_{r'} \left\{ e^{-(r^2 + r'\wedge 2)/(4s)} I_\nu(rr'/(2s)) \right\}$$

$$= \frac{e^{-x}}{2s} \left[\left(2x + \frac{\nu^2}{x} \right) I_\nu(x) - 2x I'_\nu(x) \right].$$

The angular moment multiplies each mode by $\ell_n^2/(\alpha^2 r^2) = \nu_n^2/r^2$. The direct radial and angular flat moments are both $1/(2s)$ times the direct integrand, producing the two subtractions of 1.

Finally

$$\mathrm{i}(\partial_\varphi - \partial_{\varphi'}) e^{\mathrm{i}\ell_n(\varphi - \varphi')} = -2\ell_n e^{\mathrm{i}\ell_n(\varphi - \varphi')}.$$

The direct current jet is zero for this equal-angle calculation, giving . Parity and radial endpoint symmetry give the other zero components.

For sdownarrow 0, Lemma controls the subtracted kernels and their jets. For $s \to \infty$, $e^{-m^2 s}$, together with the small-argument Bessel bounds, gives convergence of every displayed integral and fixed radial derivative. No angular sector changes in these formulas.

Remark (Required order of summation and subtraction). The angular sums in are formed before the proper-time integral. Individual unrenormalized mode integrals at coincidence need not converge. Neither a formal value of $\sum_n \ell_n$ nor termwise integration of its unregulated coincidence series is used.

One-dimensional corrections for non-Friedrichs slots

Put

$$c_\alpha = \frac{1}{4\pi^2 \alpha}.$$

Theorem (Extension corrections to all observables). *For $m > \kappa_*$, add to the Friedrichs moments*

$$\Delta F = c_\alpha \sum_n \int_m^\infty q\, c_n(q) K_{\nu_n}(qr)^2 \, \mathrm{d}q,$$

$$\Delta L = c_\alpha \sum_n \int_m^\infty \frac{q(q^2 - m^2)}{2}\, c_n(q) K_{\nu_n}(qr)^2 \, \mathrm{d}q,$$

$$\Delta R = c_\alpha \sum_n \int_m^\infty q^3 c_n(q) K_{\nu_n}'(qr)^2 \, \mathrm{d}q,$$

$$\Delta V = c_\alpha \sum_n \int_m^\infty q\, c_n(q) \frac{\nu_n^2}{r^2} K_{\nu_n}(qr)^2 \, \mathrm{d}q.$$

Here K_ν' differentiates its argument. The finite sums run only over non-Friedrichs slots. For the charged current,

$$\Delta \mathscr{J}_\varphi = -2c_\alpha \sum_n \ell_n \int_m^\infty q\, c_n(q) K_{\nu_n}(qr)^2 \, \mathrm{d}q.$$

These are absolutely convergent integrals with locally uniform convergence of every regular radial derivative. Together with , they evaluate the regular observables for the entire gapped class.

Proof. At coincidence, use polar coordinates in (ω, p) in :

$$\int_{\mathbb{R}^2} \frac{\mathrm{d}\omega\, \mathrm{d}p}{(2\pi)^2} f(\sqrt{m^2 + \omega^2 + p^2}) = \frac{1}{2\pi} \int_m^\infty q f(q)\, \mathrm{d}q.$$

The angular average of ω^2, or p^2, is $(q^2 - m^2)/2$, proving the formula for ΔL. Radial differentiation supplies $q^2 K_\nu'$^2; angular differentiation supplies ν_n^2/r^2. The antisymmetric angular jet supplies $-2\ell_n$. The convergence estimates are those in the proof of Theorem .

These formulas include $\lambda_n = 0$ and positive λ_n, for which $\kappa_* = 0$ if no other bound slot is present. They also include the stabilized logarithmic and negative-λ_n families.

Charge conjugation and domain dependence

Proposition (Symmetry with the extension transformed). *If the extension is conjugated together with the bundle, then*

$$F_{1-\beta,\overline{W}} = F_{\beta,W}, \qquad T_{1-\beta,\overline{W}} = T_{\beta,W}, \qquad \mathscr{J}_{\varphi,1-\beta,\overline{W}} = -\mathscr{J}_{\varphi,\beta,W}.$$

For the Friedrichs family this gives zero current at $\beta = 0$ and $\beta = 1/2$.

For a self-conjugate charged bundle, zero current is guaranteed for a self-conjugate domain. It is not guaranteed if two charge-conjugate radial slots are assigned different apex conditions.

Proof. For $0 < \beta < 1$, the reindexing $n \mapsto -n - 1$ negates ℓ_n and leaves ν_n unchanged. Conjugate domains have the same corresponding real coefficients c_n. The formulas for F, L, R, V are unchanged, whereas the current formulas change sign. For $\beta = 0$, use $n \mapsto -n$. At a self-conjugate domain, oddness forces the stated zero. Unequal conjugate-slot coefficients remove that cancellation.

Current representation and an exact quotient check

Let

$$f_m(\rho) = \frac{mK_1(m\rho)}{4\pi^2\rho}.$$

For an opening ratio at which $1/(2\alpha) \notin \mathbb{N}$, zero angular separation is not an image boundary. Define

$$\Xi_\beta(u) = \sum_{n \in \mathbb{Z}} \ell_n \sin(\pi\nu_n) e^{-\nu_n u}, \qquad u > 0.$$

It is understood through its convergent geometric-series expression near $u = 0$ when differentiating the Sommerfeld weight.

Proposition (Massive image–diffraction current). *Under the preceding nonboundary condition,*

$$\mathcal{J}_{\varphi,\mathrm{F},m} = -\frac{\alpha m^2}{2\pi^2} \sum_{0 < k < 1/(2\alpha)} \sin(2\pi\beta k)\cot(\pi\alpha k) K_2(2mr\sin(\pi\alpha k)) + \frac{m}{4\pi^3\alpha r} \int_0^\infty \frac{\Xi_\beta(u)}{\cosh(u/2)} K_1(2mr\cosh(u/2))\, \mathrm{d}u.$$

The all-angle definition remains ; no separate pointwise differentiation at an image boundary is asserted.

Proof. Integrating the massive image–diffraction heat formula against the Euclidean time heat factor gives the corresponding covariance formula with flat kernel f_m, as in the massless companion but with massive subtraction [12, 14]. In an angular neighbourhood containing no image boundary, the image set is constant, and its diffraction weight and one angular derivative are bounded at $u = 0$. At infinity the Bessel factor has exponential decay in

$\cosh(u/2)$. Differentiation under the integral is therefore justified.

Use

$$\frac{\mathrm{d}f_m}{\mathrm{d}(\rho^2)} = -\frac{m^2 K_2(m\rho)}{8\pi^2\rho^2}.$$

Pairing conjugate geometric images gives the first line of . The angular derivative of the diffraction weight is $\mathrm{i}\Xi_\beta$; applying $2\mathrm{i}\partial_{\varphi-\varphi'}$ gives the second line, with $\rho = 2r\cosh(u/2)$.

The formula is consistent with the established massive current literature [3]. Its derivation here fixes the particular lower-index, unit-charge, positive-meridian convention. It is not a claim of a previously unknown massive cosmic-string effect.

Proposition (Finite-character check). *If*

$$\alpha = \frac{1}{N}, \qquad \beta = \frac{p}{N}, \qquad N \geq 2, \quad 0 \leq p < N,$$

the exact Friedrichs Wick square and current are

$$F_{\mathrm{F},m} = \sum_{k=1}^{N-1} \cos\left(\frac{2\pi pk}{N}\right) \frac{mK_1(2mr\sin(\pi k/N))}{8\pi^2 r\sin(\pi k/N)},$$

$$\text{boxed}\,\mathscr{J}_{\varphi,\mathrm{F},m} = -\frac{m^2}{4\pi^2 N}\sum_{k=1}^{N-1} \sin\left(\frac{2\pi pk}{N}\right)\cot\left(\frac{\pi k}{N}\right)K_2(2mr\sin(\pi k/N)).$$

The full quotient sum includes even-N boundary images.

Proof. The finite character projection of the flat heat kernel selects covering angular modes $Nn + p$. Its angular factor and Bessel order are exactly those of ; its normalization supplies $1/\alpha = N$. Mass multiplication, Euclidean integration, and spectral continuation give the finite image covariance.

Removing the identity image and taking coincidence gives . In the image denominator,

$$\rho_k^2 = r^2 + r'^2 - 2rr'\cos\left(\frac{\varphi-\varphi'}{N} - \frac{2\pi k}{N}\right) + (\tau-\tau')^2 + (z-z')^2.$$

Apply $2\mathrm{i}\partial_{\varphi-\varphi'}$, use the derivative of f_m above, and pair conjugate phases. This gives , independently of a

proper-time coincidence summation.

The universal approach-to-threshold law

Fix an extension with $\kappa_* > 0$, and let $m\text{downarrow}\,\kappa_*$ through the gapped regime. Write

$$M_* = \sqrt{m^2 - \kappa_*^2}.$$

The critical projector kernel is

$$\Pi_*(\zeta, \zeta') = \sum_{j \in \mathscr{C}_*} \chi_j(\zeta)\overline{\chi_j(\zeta')}, \qquad \zeta = (r, \varphi).$$

It is constant in both longitudinal endpoints when viewed as a spacetime kernel.

Theorem (Logarithmic infrared residue). *For every $\mu > 0$, on sufficiently small regular Lorentzian product neighbourhoods of the diagonal,*

$$H_{W,m} = \frac{\log(\mu/M_*)}{2\pi}\Pi_* + O_{C^\infty}(1), \qquad m\text{downarrow}\,\kappa_*.$$

The remainder is bounded in every compact smooth seminorm. Changing μ changes only the bounded remainder, not the logarithmic coefficient.

In particular, with

$$d_*(r) = \sum_{j \in \mathscr{C}_*} |\chi_j(r, \varphi)|^2,$$

$$F_{W,m} = \frac{\log(\mu/M_*)}{2\pi} d_*(r) + O_{C^\infty}(1),$$

$$\mathscr{J}_{\varphi,W,m} = -\frac{\log(\mu/M_*)}{\pi} \sum_{j \in \mathscr{C}_*} \ell_{n_j} |\chi_j(r, \varphi)|^2 + O_{C^\infty}(1).$$

The Wick square diverges at every regular point, since $d_*(r) > 0$. The current's logarithmic coefficient may cancel.

Proof. The transverse resolvent has, at $q = \kappa_*$, the spectral pole

$$(H_W + q^2)^{-1} = \frac{\Pi_*}{q^2 - \kappa_*^2} + \mathscr{R}_{\mathrm{reg}}(q).$$

For the explicit formulas, the same residue follows directly from

$$c_{n}(q)K_{\nu_{n}}(qr)K_{\nu_{n}}(qr^{'}) \sim \frac{N_{j}^{2}K_{\nu_{j}}(\kappa_{*}r)K_{\nu_{j}}(\kappa_{*}r^{'})}{q^{2}-\kappa_{*}^{2}}.$$

The logarithmic coefficient has the analogous $\nu = 0$ limit.

In the Euclidean correction , let $k = (\omega, p)$. Near $k = 0$, $q^{2} - \kappa_{*}^{2} = M_{*}^{2} + |k|^{2}$. Choose a smooth compact k-cutoff equal to one near zero. Subtract from the pole integral its endpoint value of the exponential. All nonconstant terms are $O(|k|)$, and hence are uniformly integrable against $(M_{*}^{2} + |k|^{2})^{-1}\,\mathrm{d}^{2}k$. After angular integration their zero-order constant part is

$$\int_{\mathbb{R}^{2}} \frac{\chi(k)\quad \mathrm{d}^{2}k}{|k|^{2}+M_{*}^{2}(2\pi)^{2}} = \frac{1}{2\pi}\log\left(\mu/M_{*}\right) + O(1).$$

Differentiating endpoint variables does not change the conclusion: derivatives of Π_{*} retain the same logarithm, and derivatives of the longitudinal exponential supply powers of k.

After subtracting , the remaining low-frequency transverse resolvent is uniformly regular. The large-frequency correction retains the exponential Bessel bounds of Theorem , uniformly for m near the positive number κ_{*}. These estimates hold in a fixed small complex longitudinal neighbourhood. They prove both the C^{∞} Euclidean assertion and its local Lorentzian continuation. The Friedrichs remainder is bounded there as $m \to \kappa_{*} > 0$.

Taking coincidence gives . Applying the angular current jet to $e^{\mathrm{i}\ell_{n_{j}}(\varphi-\varphi^{'})}$ gives $-2\ell_{n_{j}}$, proving . Every nonzero critical Bessel bound function has strictly positive modulus at every $r > 0$; therefore $d_{*}(r) > 0$.

To state the stress coefficient explicitly, put

$$R_{*}(r) = \sum_{j\,\in\,\mathscr{C}_{*}} |\partial_{r}\chi_{j}|^{2}, \qquad V_{*}(r) = \sum_{j\,\in\,\mathscr{C}_{*}} \frac{\nu_{j}^{2}}{r^{2}}|\chi_{j}|^{2}, \qquad D_{*} = \mathscr{L}_{r}d_{*}.$$

Define the mixed tensor

$$\Theta^{t}_{\ t} = \Theta^{z}_{\ z} = n_{\mathrm{f}}\Big(\xi - \frac{1}{4}\Big)D_{*},$$

$$\Theta^{r}{}_{r} = n_{\mathrm{f}}\Big[R_{*} - \xi d_{*}'' + \Big(\xi - \frac{1}{4}\Big)D_{*}\Big],$$

$$\Theta^{\varphi}{}_{\varphi} = n_{\mathrm{f}}\Big[V_{*} - \xi d_{*}'/r + \Big(\xi - \frac{1}{4}\Big)D_{*}\Big].$$

Corollary (Stress residue and impossibility of local removal). *As m*downarrow κ_{*},

$$T^{\mu}{}_{\nu} = \frac{\log(\mu/M_{*})}{2\pi}\Theta^{\mu}{}_{\nu} + O_{C^{\infty}}(1).$$

The tensor Θ is conserved and satisfies

$$\Theta^{\mu}{}_{\mu} = 3\Big(\xi - \frac{1}{6}\Big)n_{\mathrm{f}}D_{*} - n_{\mathrm{f}}\kappa_{*}^{2}d_{*}.$$

The divergence in cannot be removed by the local freedoms , by a current background counterterm, or by an apex-supported contact.

Proof. Apply the stress bidifferential operator to . The projector is independent of t, z, so its L-moment is zero, while its radial and angular moments are R_{*}, V_{*}. Replacing m by κ_{*} in the leading coefficient changes only bounded terms, since $(m^{2} - \kappa_{*}^{2})\log M_{*} \to 0$. This proves –. The projector is a threshold massive bisolution, so Theorem proves its identities.

The Wick residue is a nonconstant, extension-dependent profile made from $K_{\nu_j}(\kappa_{*}r)^{2}$, nonzero on regular open sets. A local flat counterterm is spacetime constant and cannot cancel it. An apex-supported distribution vanishes on all regular tests. The allowed current ambiguity already vanishes on this host.

Remark (An infrared law, not a new ultraviolet prescription). One can subtract the projector logarithm as a formal infrared finite part. That operation is extension dependent and nonlocal in the transverse direction. It is not the allowed local ultraviolet subtraction, and it does not establish positivity of an unrestricted threshold state. Theorem remains the state verdict.

Massless, flat, and threshold limits

Friedrichs massless limit

Theorem (Compact-regular massless convergence). *For fixed* $0 < \alpha < 1$, $0 \le \beta < 1$, *and fixed* ξ, *the Friedrichs covariance converges to its massless spectral covariance in* $\mathscr{D}'(M \times M)$ *as* $m \to 0$.

The renormalized Euclidean remainders converge in C^∞ *on sufficiently small compact regular developing product neighbourhoods. In particular, the functions* $F, L, R, V, T^\mu{}_\nu, \mathscr{J}_\varphi$ *and every fixed regular radial derivative converge uniformly on compact subsets of* $r > 0$. *Their limits are*

$$F_{\mathrm{F},0} = \frac{P_\beta}{48\pi^2 r^2}, \qquad P_\beta = \alpha^{-2}(1 - 6\beta(1-\beta)) - 1,$$

$$T^\mu{}_{\nu,\mathrm{F},0} = \frac{n_{\mathrm{f}} C_\beta}{1440\pi^2 r^4}\operatorname{diag}(1,1,-3,1) + n_{\mathrm{f}}\left(\xi - \frac{1}{6}\right)\frac{P_\beta}{48\pi^2 r^4}\operatorname{diag}(4,-2,6,4),$$

$$C_\beta = \alpha^{-4}(1 - 30\beta^2(1-\beta)^2) - 1,$$

$$\mathscr{J}_{\varphi,\mathrm{F},0} = \frac{\beta(1-\beta)(2\beta-1)}{12\pi^2\alpha^2 r^2}.$$

Proof. For large proper time on a compact regular set, the spatial massless heat kernel is $O(s^{-3/2-\nu_*})$, where $\nu_* = \min(\beta, 1-\beta)/\alpha$. The Euclidean time factor gives $O(s^{-2-\nu_*})$. Small-argument Bessel expansions give integrable large-s bounds after every fixed regular endpoint derivative. At small s, Lemma provides a mass-uniform bound for the subtracted kernel and all such derivatives. Dominated convergence proves the stated Euclidean and observable convergence.

On compact spacetime tests, the same massless infrared estimate gives $\operatorname{dom}(B_{\mathrm{F},0}^{-1/4})$. Spectral domination by the massless weight and continuity of $e^{-\mathrm{i}T\sqrt{\lambda+m^2}}/\sqrt{\lambda+m^2}$ away from $\lambda = 0$ prove distributional covariance convergence.

For clarity, Appendix evaluates the needed massless diagonal jets directly. They give –, matching the normalizations of I-B-VP and I-B-VP-0 [13, 14].

The term $a_{\mathrm{ren}} m^2$, if retained with a fixed finite coefficient, disappears in this limit; the stress term $b_{\mathrm{ren}} m^4$ does likewise. No differentiability in the mass parameter at zero is inferred from compact-regular convergence.

Flat trivial-holonomy limit

Proposition (What vanishes in the flat limit). *For the Friedrichs family at fixed $m > 0$,*

$$(\alpha, \beta) \to (1, 0)$$

gives

$$F_{\mathrm{F},m}, \quad T_{\mathrm{F},m}, \quad \mathscr{J}_{\mathrm{F},m} \to 0$$

with all fixed radial derivatives on compact subsets of $r > 0$.

The same conclusion is false for a general retained non-Friedrichs extension.

Proof. At $(\alpha, \beta) = (1, 0)$,

$$\sum_{n \in \mathbb{Z}} I_{|n|}(x) = e^x, \qquad \sum_{n \in \mathbb{Z}} n I_{|n|}(x) = 0.$$

The direct flat heat kernel is therefore the entire kernel, so all subtracted moments vanish.

Convergence on compact regular sets follows either from the localized heat bounds and dominated parameter convergence, or from the covariance image–diffraction representation. For α sufficiently close to 1, its nonidentity geometric diagonal sum is empty, and the diffraction denominator at zero angular separation stays away from its transition. The massive Bessel factor dominates its large-u tail uniformly; all fixed radial jets therefore converge to zero.

For a counterexample to unrestricted extension vanishing, retain the untwisted logarithmic length L and take $m > \kappa_L$. At $\alpha = 1$, $\beta = 0$, gives

$$\Delta F(r) = \frac{1}{4\pi^2} \int_m^\infty \frac{q K_0(qr)^2}{\log(q/\kappa_L)} \mathrm{d}q > 0.$$

The cone has become flat, but the point-defect domain has not become Friedrichs.

The threshold limit is not the massless limit

For a fixed extension with $\kappa_* > 0$, the admissible ordinary massive family ends at $m = \kappa_*$. It cannot be continued to $m = 0$ through ordinary nonnegative spectral ground states. Theorem shows a divergent logarithm at that endpoint, whereas Theorem gives a finite Friedrichs massless limit.

The geometric arrival threshold of I-B-G, often denoted T_{arr}, is a travel-time condition. The spectral threshold $m = \kappa_*$ is an energy condition. Neither can be substituted for the other.

Finite reporting, scope, and the regime ledger

Finite Enter reports

A finite supplied report may install encodings of

$$(\alpha, \beta, \xi, m, \{\lambda_n, L_n\}, \kappa_*, \text{regime}, \text{normalization}, \text{formula identifiers}).$$

Exact real parameters need not have exact finite-bit encodings; a numerical report must specify its parameter enclosures.

For a calibrated finite clock basis and all surviving registers A, define

$$D_j = \langle \tau_j|_C \otimes I_A, \qquad \mathscr{M}(X) = \sum_j \mathrm{lvert} j, w_j \rangle \mathrm{langle} j, w_j|_R \otimes D_j X D_j^\dagger.$$

Then

$$\sum_j D_j^\dagger D_j = I_{CA}, \qquad \mathrm{Tr}_R\, \mathscr{M}(X) = \mathrm{Tr}_C\, X,$$

including arbitrary untouched finite references. The equalities follow by expanding X in clock matrix units, exactly as in I-B, En09-A, and XVI-O1-ST [11, 17, 18]. The clock is consumed; the surviving A is not.

This reports installed descriptions. It is not a field acquisition map and does not evaluate an unknown continuum covariance by destructive readout.

Regular-part tests annihilate apex-supported contacts, but they do not annihilate the functions $\chi_j(r, \varphi)$. Tests

restricted by ignore only the relevant zero-frequency critical moments, not every piece of the channel. Tests orthogonal to the entire channel ignore every modification confined to it. These are different restrictions.

Finite-test coverage remains class relative. Vanishing of finitely many evaluations is a continuum identity only after an independently justified defect space and a separating evaluation map are supplied [19]. It cannot repair the divergent positive quadratic form in Theorem .

Evidence

The displayed analytical formulas and proofs are the claimed evidence. No numerical execution or raw massive-current runs are supplied.

We retain the labels , , and for any future report [18]. Such a report must retain at least:

source/version, domain convention, mass and gap encodings,
current sign/component, test or radius, subtraction scheme,
raw values, precision, cutoffs, tail bounds, input bounds,
residuals, and actual evidence status.

A rounded estimate with $m - \kappa_*$ unresolved does not certify a gapped regime. Near threshold, input uncertainty in that difference is a separate error, not a quadrature remainder.

Regime ledger

begingroup 1.12 toprule Extension and mass State status Observable status

midrule endfirsthead toprule Extension and mass State status Observable status

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Friedrichs, $m > 0$ Ordinary spectral Fock ground state All regular one-point functions evaluated by – and ; all opening ratios.

Non-Friedrichs with no transverse bound state, $m > 0$ Ordinary gapped spectral state Explicit convergent corrections –.

Logarithmic or negative-power bound slots, $m > \kappa_*$ Mass-stabilized ordinary spectral state Same correction

formulas; local massive Hadamard subtraction and Ward identities hold.

Critical bound slots, $m = \kappa_* > 0$ No ordinary compact-test spectral state; restricted state on $\mathscr{D}_*$ No unrestricted field-square/stress/current package is assigned at threshold. The gapped family's logarithmic residues are explicit.

$m < \kappa_*$ Negative band; original nonnegative-square-root prescription excluded No original-field spectral vacuum observables assigned by this paper. Projected substitutes are different constructions.

Friedrichs, $m = 0$ Massless state retained as a limit The companion Wick square, stress, and current are recovered in the compact-regular topology.

endgroup

Boundaries and programme status

The calculation extends neither the state nor its field algebra across $\mathscr{W}$. It defines no product of the conical connection with apex-supported distributions. The massless finite contact dimensions in I-B-VP-1 are not imported: a massive apex extension problem has additional scales and requires a new classification [15].

The local Hadamard theorem is not a new massive global wavefront theorem. The exact massless result of I-B-VP-2 retains its stated scope [16]. Thermal states, spinors, gauge constraints, finite cores, and interactions remain outside this paper.

A c-number conserved stress is not an Einstein source law. The m^2 Wick ambiguity is not a solution of the vacuum-energy problem. The threshold obstruction is not a cosmological bounce, Page curve, de Sitter construction, or removal of the external geometric time coordinate.

Nothing here discharges XVI-N, Route B, or a joint characteristic-product obligation. The historical H2 record, XIV-I's constructed I2 internal-amplitude result, and the unresolved full compositions at I3/J2 are not updated or conflated [20]. The downstream XV-B quantum-admission exit is not opened.

No version-specific O.1 status is changed. An O.1= entry remains so outside any independently earned scope; this manuscript neither verifies nor revokes a separately reported subsector closure. is not promoted. No OpenAI

catalogue entry supplies a spectral or Hadamard premise.

Conclusion. For the declared product-extension class, the massive problem has a precise endpoint: an ordinary spectral ground state and all three regular observables exist above threshold; the critical bound-channel moments identify exactly why the ordinary covariance fails at threshold; and a negative band excludes the prescription below it. The one-dimensional extension integrals and the projector logarithm make that statement quantitative. The achievement is an extension-resolved free-field construction on a prescribed spacetime, not a selection of its geometry, apex physics, or gravitational dynamics.

appendix

Massless diagonal jets and normalization checks

This appendix evaluates the massless limits used in Theorem . It does not assign values to divergent coincidence mode sums.

For Euclidean separation set

$$\cosh\eta = \frac{r^2 + r'^2 + (\tau - \tau')^2 + (z - z')^2}{2rr'}.$$

Heat integration gives the convergent separated kernel

$$G^E_{\mathrm{F},0} = \frac{1}{8\pi^2 \alpha r r' \sinh\eta} \sum_{n \in \mathbb{Z}} e^{-|\ell_n|\eta/\alpha} e^{i\ell_n(\varphi - \varphi')}.$$

At zero angular separation, $a = \eta/\alpha > 0$,

$$\sum_n e^{-|\ell_n|a} = \frac{\cosh((1/2 - \beta)a)}{\sinh(a/2)} = \frac{2}{a} + B_2(\beta)a + \frac{B_4(\beta)}{12}a^3 + O(a^5),$$

where

$$B_2(\beta) = \frac{1}{6} - \beta(1 - \beta), \qquad B_4(\beta) = \beta^2(1 - \beta)^2 - \frac{1}{30}.$$

Subtracting the direct flat kernel gives

$$F_{\mathrm{F},0} = \frac{P_{\beta}}{48\pi^{2}r^{2}},$$

and its longitudinal second jet gives

$$L_{\mathrm{F},0} = \frac{1}{\pi^{2}r^{4}}\left(\frac{C_{\beta}}{1440} + \frac{P_{\beta}}{144}\right).$$

Inserting this into gives the time component of . Radial conservation and the trace identity then give the radial and angular entries: a homogeneous diagonal stress proportional to r^{-4} has $T^{\varphi}{}_{\varphi} = -3T^{r}{}_{r}$. This proves the displayed full stress.

For the current define the absolutely convergent sum

$$A_{\beta}(a) = \sum_{n} \ell_{n} e^{-|\ell_{n}|a}.$$

Splitting its positive and negative modes gives

$$A_{\beta}(a) = -\frac{\mathrm{d}}{\mathrm{d}a}\frac{\sinh((1/2-\beta)a)}{\sinh(a/2)} = \frac{2}{3}B_{3}(\beta)a + O(a^{3}),$$

$$B_{3}(\beta) = \frac{1}{2}\beta(1-\beta)(1-2\beta).$$

Use a nonzero longitudinal split before taking the angular derivative of the kernel. The regulated current is

$$-\frac{A_{\beta}(\eta/\alpha)}{4\pi^{2}\alpha r^{2}\sinh\eta}.$$

Its regular coincidence limit is

$$-\frac{B_{3}(\beta)}{6\pi^{2}\alpha^{2}r^{2}},$$

which is .

As a separate quotient check, $K_{2}(z) \sim 2z^{-2}$ in gives

$$\mathcal{J}_{\varphi,\mathrm{F},0} = -\frac{1}{8\pi^{2}Nr^{2}}\sum_{k=1}^{N-1}\sin\left(\frac{2\pi pk}{N}\right)\cot\left(\frac{\pi k}{N}\right)\csc^{2}\left(\frac{\pi k}{N}\right).$$

The finite identity

$$\sum_{k=1}^{N-1} \sin\left(\frac{2\pi pk}{N}\right)\cot\left(\frac{\pi k}{N}\right)\csc^{2}\left(\frac{\pi k}{N}\right) = \frac{2}{3}p(N-p)(N-2p)$$

recovers the same coefficient. It follows by taking second differences in p, using finite root-of-unity sums, and the endpoint values at $p = 0, N$, as in I-B-VP-0 [14].

Acceptance audit

toprule Potential error Correction used here

midrule endfirsthead toprule Potential error Correction used here

midrule endhead bottomrule endfoot

Calling every negative transverse mode unstable at positive mass Use the full product band and $m^2 - \kappa_*^2$, not the transverse sign alone.

Omitting non-Friedrichs domains with no bound state Positive and zero λ_n are included for every $m > 0$.

Inferring covariance finiteness from $B \geq 0$ Use the gap estimate, or the exact critical-channel integral and its moment restriction.

Treating absence of full L^2 eigenvalues as absence of an infrared obstruction The critical divergence is continuous longitudinal spectrum at $p = 0$.

Using an unspecified companion Krein coefficient Derive – in the displayed boundary convention.

Claiming the Lorentzian extension correction is globally smooth Only Euclidean analyticity and near-diagonal Lorentzian analyticity are proved.

Subtracting only the massless pole Use the full massive direct kernel, including its logarithmic short-distance terms.

Listing only the m^2 Wick ambiguity Retain the $m^4 g_{\mu\nu}$ stress ambiguity before Minkowski normalization.

Claiming smooth mass dependence through zero while normal ordering at every mass Prove compact-regular limits; do not claim analytic mass dependence at zero in a fixed-scale family.

Erasing the current with a real-part replacement Use the complex antisymmetric first jet.

Integrating divergent coincidence modes separately Sum angular modes and perform the direct subtraction before proper-time integration.

Asserting every extension has a trivial flat limit A retained logarithmic point-defect domain gives an explicit nonzero counterexample.

Removing the threshold logarithm with an apex contact Its residue is nonzero on regular open sets.

Calling a restricted threshold state the original local scalar vacuum State its moment-constrained test algebra explicitly.

Confusing a massive mass shell with an ordinary wavefront set The principal characteristic cone remains null; global massive microlocal analysis is not claimed.

Promoting a supplied finite report to acquisition No continuum field input or detector-field coupling is present in the report instrument.

Declarations

Domain. All quantum states and one-point observables discussed as ordinary fields have regular endpoints $r > 0$. The threshold construction is explicitly restricted to $\mathscr{D}_*$. No field algebra at $\mathscr{W}$ is defined.

Dependency and source scope. The geometry and companion conventions are stated in the text. The Bessel resolvent, gapped state estimate, threshold restriction, one-dimensional corrections, and logarithmic residue are proved here. Only the supplied I-B-VP-v4 excerpts, not an unavailable full proof of its named Krein theorem, are attributed. Other companion comparisons do not transfer their equation spaces, products, or admission conclusions.

Novelty. The contribution is a completed simultaneous construction for the declared companion target, with an explicit threshold residue and observable diagnosis. No priority is claimed for massive cosmic-string currents, Bessel extensions, or the infrared problem of a massless one-dimensional momentum channel.

Verification. This source is a single pdfLaTeX document with an internal bibliography. No successful compiler run, numerical experiment, independent rebuild, proof-assistant verification, independent referee review, or journal acceptance is asserted.

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